Some Classes of Spirallike Functions

By

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Declaration

I, Nazar Khan, registration number **CIIT/FA12-PMT-005/ISB,** hereby declare that I have produced the work presented in this thesis during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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Certificate

It is certified that Nazar Khan bearing CIIT/FA12-PMT-005/ISB has carried out all the work related to this thesis under my supervision at the Department of Mathematics COMSATS Institute of Information Technology Islamabad and the work fulfills the requirement for award of PhD degree.

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This work is dedicated

to

my parents, my teachers

And

my famil

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ABSTRACT

Some Classes of Spirallike Functions

In this thesis, we define some new subclasses of analytic spirallike functions using the techniques of differential subordination, convolution and the concepts of conic domains, bounded boundary rotation and bounded radius rotation. We study these new classes thoroughly and investigate several coefficient results, radii problems, convolution preserving properties, inclusion results, integral preserving mapping properties along with various other useful applications. Most of these results are sharp. Various known and new results are also derived as special cases from our results.

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LIST OF ABBREVIATIONS

$\mathbb{C}$	Set of complex number
$\mathbb{E}$	Open unit disk
$k(z)$	Koebe function
$L_0(z)$	Möbius function
$(\mathcal{E})_n$	Pochhammer symbol
$f * g$	Convolution of f and g
$\prec$	Subordination
$\mathcal{A}$	Class of normalized analytic functions
$\mathcal{S}$	Class of univalent functions
$\mathcal{A}(p)$	Class of multivalent functions
$\mathcal{P}$	Class of Carathéodory functions
$\mathcal{C}$	Class of Convex functions
$\mathcal{C}(\lambda)$	Class of Convex functions of order λ
$\mathcal{M}(\alpha)$	Class of α $-$convex functions
$\mathcal{S}^*$	Class of Starlike functions
$\mathcal{S}^*(\lambda)$	Class of Starlike functions of order λ
$\mathcal{S}_\beta$	Class of β $-$Spirallike functions of type β
$\mathcal{S}_\beta(\lambda)$	Class of β $-$Spirallike functions of order λ
$\mathcal{K}$	Class of Close to Convex functions
$S^*{}_s$	Class of Starlike functions with respect to symmetric points

$\mathcal{R}_m$	Class of analytic functions with bounded radius rotation
$\mathcal{V}_m$	Class of analytic functions with bounded boundary rotation
$\mathcal{UCV}$	Class of uniformly convex functions
$\mathcal{ST}$	Class of uniformly starlike functions
$k-\mathcal{UCV}$	Class of k-uniformly convex functions
$k-\mathcal{ST}$	Class of k-uniformly starlike functions
$\mathcal{P}[\mathbb{A},\mathbb{B}]$	Class of Janowski functions
$\mathcal{C}[\mathbb{A},\mathbb{B}]$	Class of Janowski Convex functions
$\mathcal{S}^*[\mathbb{A},\mathbb{B}]$	Class of Janowski Starlike functions
Ω_k	Conic domain
$\Omega_{k,\lambda}$	Generalized Conic domain
$\Omega[\mathbb{A},\mathbb{B}]$	Circular domain
$\mathbb{k}(a,c)$	Carlson - Shaffer operator

Chapter 1

Introduction

Geometric Function Theory is a branch of complex analysis which deals with the geometric properties of analytic functions. The men who worked on it firstly were Cauchy, Riemann and Weierstrass. Riemann showed that there always exists a unique analytic function f which maps any given simply connected domain $\mathfrak{D}_1 \neq \mathbb{C}$ with atleast two boundary points in the z-plane onto a simply connected $\mathfrak{D}_2$ in the w-plane. This theorem is generally recognized as one of the most fundamental contributions in this area. The foundation stone of Geometric Function Theory is the theory for univalent functions, commenced by Koebe [24] in 1907. The collection of all functions that are analytic and univalent in $\mathbb{E} = \{\mathrm{z} : \mathrm{z} \in \mathbb{C}, |\mathrm{z}| < 1\}$ and satisfying $f(0) = f'(0) - 1 = 0$ is named as $\mathcal{S}$. The well known subclasses of the class $\mathcal{S}$ are $\mathcal{C}$ and $\mathcal{S}^*$ of convex and starlike univalent functions respectively have been of major significance. In 1915, Alexander [2] gave a beautiful relation between the class $\mathcal{C}$ and $\mathcal{S}^*$, that is $f \in \mathcal{C}$ iff $\mathrm{z}f' \in \mathcal{S}^*$. Kaplan [37] investigated the class $\mathcal{K}$ of close to convex functions and proved that these functions are univalent. Then a new subclass of $\mathcal{S}$ known as class of quasi convex functions denoted by $\mathcal{C}^*$ was introduced and studied in [65, 69]. The class $\mathcal{C}^*$ has the same relation with the class $\mathcal{K}$ as $\mathcal{C}$ has with $\mathcal{S}^*$.

In 1933, Spacek [118, 27] extended the notion of starlike functions and initiated the class $\mathcal{S}_\beta$, $|\beta| < \frac{\pi}{2}$, the class of $\beta-$spirallike functions as follows.

An analytic function f belong to the class $\mathcal{S}_\beta$, iff

$$\mathfrak{Re}\left(e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right) > 0, \qquad \mathrm{z} \in \mathbb{E}. \tag{1.1}$$

The functions which satisfy (1.1) are called spirallike functions of type β. For $\beta = 0$ in (1.1), it can easily be seen that this class coincides with the class $\mathcal{S}^*$, the class of starlike univalent functions. In 1967, Libera [44] extended this definition to functions spirallike of order λ, denoted by $\mathcal{S}_\beta(\lambda)$. Later, in 1969 Robertson considered the class $\mathcal{C}_\beta$ of analytic functions defined as $f \in \mathcal{C}_\beta$ iff $\mathrm{z}f' \in \mathcal{S}_\beta$ and further it was studied by Libera and Zeigler

[45] and Chichra [8] . Many important features of some classes of spirallike functions were studied by several authors together with Silvia [116], Libera [44], Keogh and Merkes [41], Uyanik et.al. [126], Ravichandran et.al. [100], Noor and Bukhari [70] and recently by Kim and Sugawa [39].

As generalization of univalent functions, multivalent (p-valent) analytic functions have been discussed, see [17, 24, 27]. A function f analytic in a domain $\mathfrak{D} \subset \mathbb{C}$ is called multivalent (p-valent) function, see [27], $p \in \mathbb{N} = \{1, 2, 3, ...\}$ in $\mathfrak{D}$ if for every complex number z_0, the equation $f(\mathrm{z}) = \mathrm{z}_0$ does not have more than p roots in $\mathfrak{D}$ and there exists a complex number z_1 such that the equation $f(\mathrm{z}) = \mathrm{z}_1$ has exactly p roots in $\mathfrak{D}$. Many researchers, see for example [17, 24, 27], have studied certain subclasses of multivalent functions such as convex and starlike.

The set of functions with positive real part denoted by $\mathcal{P}$, called Caratheodary functions [6] participate a key role in this area of research and the majority of the subclasses of univalent functions are associated with them. For the criteria for convexity of univalent functions, the class $\mathcal{P}$ was employed by Study [121] in 1913. In 1921, Nevalinna [62] connected the class of starlike function by means of class $\mathcal{P}$.

The idea of functions with bounded boundary rotation was set up by Lowner [52], in 1917. Then Paatero [92] refined and developed this concept in more systematic way. We symbolize, $\mathcal{V}_m$, $m \geq 2$, by the class of said functions. The class $\mathcal{R}_m$, $m \geq 2$ of functions with bounded radius rotation was initiated by Tammi in 1952, see [122]. Remarkable research articles were written by numerous authors like Kirwan [42], Brannan [5], Pinchuk [97] and Noor [67, 68, 71, 72] in the development of this area of Geometric Function Theory. The class $\mathcal{P}_m$, $m \geq 2$ was introduced by Pinchuk [97] and he gave the criterion for functions to be form $\mathcal{V}_m$ or $\mathcal{R}_m$. Using the concept of $\mathcal{P}_m$, the classes $\mathcal{V}_m$ and $\mathcal{R}_m$ can be defined as:

$$\begin{aligned} \mathcal{V}_m &= \left\{ f \in \mathcal{A} : \left(\frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})} \right) \in \mathcal{P}_m \right\}, \\ \mathcal{R}_m &= \left\{ f \in \mathcal{A} : \left(\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} \right) \in \mathcal{P}_m \right\}. \end{aligned}$$

If the functions f and g both are analytic in $\mathbb{E}$, then f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w such that $f(\mathrm{z}) = g(w(\mathrm{z}))$. Janowski [30] introduced and studied the class $\mathcal{P}[\mathbb{A}, B]$ by using the concept of subordination as follows. The class $\mathcal{P}[\mathbb{A}, \mathbb{B}]$, $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$ contain the functions h and satisfying the property

$$h(\mathrm{z}) \prec \frac{1+\mathbb{A}\mathrm{z}}{1+\mathbb{B}\mathrm{z}}, \quad \mathrm{z} \in \mathbb{E}.$$

In 1991, Goodman [22, 23] defined subclasses of $\mathcal{C}$ and $\mathcal{S}^*$ by introducing the concepts of uniformly convex (starlike) functions. These classes were defined by their geometrical mapping properties with image domains associated with conic regions. Soon after, Ronning [107], Ma and Minda [53] independently gave a most suitable form of Goodman criteria of these classes and developed the parabolic domain. The one variable characterization of the classes $\mathcal{UCV}$ and $\mathcal{ST}$ are defined as.

$$\mathcal{UCV} = \left\{ f \in \mathcal{A} : \mathfrak{Re}\left[1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right] > \left|\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right|, \qquad \mathrm{z} \in \mathbb{E} \right\},$$

and

$$\mathcal{ST} = \left\{ f \in \mathcal{A} : \mathfrak{Re}\left[\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right] > \left|\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}-1\right|, \qquad \mathrm{z} \in \mathbb{E} \right\}.$$

This one variable characterization gave birth to the first conic (parabolic) domain.

Finally, in 1999, Kanas and Wisniowska [33, 34] generalized the parabolic domain and introduced the general conic domain which represents all the three conic structures, that is parabola, hyperbola and ellipse. A huge number of well-known mathematicians like Srivastava, Noor, Owa etc studied these conic domains in detail, see [36, 85].

In this thesis, we use the concepts of differential subordination, spirallike functions of type β and conic domains, to define various new classes of multivalent and univalent functions in $\mathbb{E}$. Further, we focus on the study of subclasses of analytic spirallike functions. We shall study and investigate some basic properties of these classes such as inclusion relations, radius problems, convolution properties, integral preserving properties, sufficiency criteria, Fekete-Szego inequality and some other problems. A concise introduction of all chapters is given below.

Chapter 2 accentuate on some preliminary concepts of Geometric Function Theory and definitions which are used in the successive chapters. The main tools of our work, subordination and convolution are briefly discussed here. Some lemmas, are included which will be of use in subsequent chapters. We mention here that no new definitions or results are included in this chapter and all the contents are known and the relevant references are given.

In Chapter 3, using the concept of subordination we define a new class $k\text{–}\mathcal{UM}^*(p, \alpha, \beta, \gamma)$, for α, β real, $|\beta| < \frac{\pi}{2}$ and $-\frac{1}{2} \leq \gamma < 1$ of analytic functions related with conic domains. The class $k\text{–}\mathcal{UM}^*(p, \alpha, \beta, \gamma)$ generalize the class $\mathcal{SC}(\alpha, \beta)$, the class of α-spiral convex functions defined by Umarani [124]. Classes $k\text{–}\mathcal{U}\mathrm{R}^*(p, \beta)$ and $k\text{–}\mathcal{UC}(p, \beta)$ are also introduced in this chapter by using the idea of spirallike and Robertson functions. Many interesting properties such as inclusion results, sufficient condition and Fekete-Szego inequality for these classes of functions are investigated here and many known results are deduced from our main results as special cases.

In Chapter 4, we define new class $k\text{–}\mathcal{US}_s^\beta[\mathbb{A}, \mathbb{B}]$, β is real, $|\beta| < \frac{\pi}{2}$, $k \geq 0$, $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$ and the corresponding class $k - \mathcal{UC}_s^\beta[\mathbb{A}, \mathbb{B}]$ by using the idea of the classes $\mathcal{S}^*$ and $\mathcal{C}$ with respect to symmetrical points associated with generalized conic domain $\Omega_k[\mathbb{A}, \mathbb{B}]$. These generalized classes contain many known classes by varying the values of parameters. We have proved the convolution properties and Fekete-Szego inequality for the functions of these classes.

In Chapter 5, we define $\mathcal{M}_m^*(\alpha, \beta)$, the class of generalized spirallike Mocanu variation by making use of the classes $\mathcal{V}_m$ and $\mathcal{R}_m$. The class $\mathcal{M}_m^*(\alpha, \beta)$ contains the class $\mathcal{S}_\alpha^\beta$ defined by Silvia [116]. This chapter also contains the classes of Mocanu variation and β-spirallike functions of order λ as special cases. We define $\mathcal{R}_m^*(\beta)$ and $\mathcal{B}_m(\alpha, \beta, \gamma)$ and establish the relations with various known classes. Some interesting properties of these classes including inclusion results, arc length problem and a sufficient condition for univalency are studied. The contents of this chapter are published in Filomat, see [82].

In Chapter 6, we define the class $\mathcal{SC}_p^{\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \alpha)$, $\mathfrak{b} \in \mathbb{C} \diagdown \{0\}$, λ is real with $|\beta| < \frac{\pi}{2}$, $\alpha > 1$, by using the reciprocal order and the operator $\mathbb{k}_p$ defined by Saitoh [111, 112] in multivalent functions. Some interesting properties of this class, such as coefficient estimates, sufficiency criteria, Fekete-Szego inequality and inclusion result are studied. We also prove that the class $\mathcal{SC}_p^{\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \alpha)$ is closed under some integral operator.

In Chapter 7, we have generalized the class $\mathcal{R}$ by using pth derivative of analytic functions and introduce the class $\mathcal{R}_p$. This generalized class contains many known classes. We have proved convolution properties, integral preserving properties and the closure property with respect to Hadamard product for the class $\mathcal{R}_p$. We also prove that the ℓth partial sum of $f \in \mathcal{R}_p$, $S_\ell^p(\mathrm{z}, f)$ is univalent in $\mathbb{E}$. The contents of this chapter are published in Maejo Int. J. Sci. Technol, see [83].

Chapter 2

Some Preliminary Concepts

This chapter has goals to provide some basic concepts of Geometric function Theory. It will start with analytic and univalent functions. The class $\mathcal{S}$ of normalized analytic univalent functions and its basic subclasses will be discussed. The main focus and theme of concern of our work is the spirallike functions of type β and its generalizations. The functions of radius rotation and bounded boundary rotation are also argued. We also discuss conic domains with some interesting properties. For the sake of completeness, we include and explore various classical results. Some of them are proved here and for the proof of others, we refer standard texts, for example, see [17, 24, 27]. At the end, there will be some preliminary well known results to be used in upcoming chapters.

2.1 Analytic and Univalent Functions

We here state the class $\mathcal{A}$ of normalized analytic functions in $\mathbb{E} = \{\mathrm{z} : \mathrm{z} \in \mathbb{C}, |\mathrm{z}| < 1\}$. Also we give the definition of the class $\mathcal{S}$ and some of its properties.

Definition 2.1.1. A complex valued function f is analytic at z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

Theorem 2.1.1. Let $\mathfrak{D} \subset \mathbb{C}$ be a simply connected domain with at least two boundary points. Then there is a unique analytic function f which maps $\mathfrak{D}$ conformally onto the open unit disk $\mathbb{E}$. See [17].

Definition 2.1.2. A function f is in the class $\mathcal{A}$, if it is analytic in $\mathbb{E}$ and is normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus a function $f \in \mathcal{A}$ has a Taylor's series expansion of the form

$$f(\mathrm{z}) = \mathrm{z} + \sum_{\ell=2}^{\infty} a_\ell \mathrm{z}^\ell, \qquad \mathrm{z} \in \mathbb{E}. \tag{2.1.1}$$

Univalent Functions

A function f is univalent in $\mathbb{E}$ if it presumes no value more than once in $\mathbb{E}$. That type of functions are also called simple or schlicht in $\mathbb{E}$. If f is univalent in $\mathbb{E}$ then the domain $\mathfrak{D} = f(\mathbb{E})$ is a simple (schlicht) domain. In 1909, Koebe [40] gave the basic idea about the univalent functions.

Definition 2.1.3. A function f is univalent in $\mathbb{E}$, if

$$\mathrm{z}_1 \neq \mathrm{z}_2 \quad \text{implies that} \quad f(\mathrm{z}_1) \neq f(\mathrm{z}_2), \quad \mathrm{z}_1, \mathrm{z}_2 \in \mathbb{E}.$$

For analytic function f the condition $f'(\mathrm{z}_0) \neq 0$ is equivalent to the local univalence at z_0 that is, f is univalent in some neighborhood of z_0.

We are mostly concerned in univalent functions that are also analytic in $\mathbb{E}$.

The class $\mathcal{S}$ of normalized univalent functions will be defined as follows.

Definition 2.1.4. The class $\mathcal{S}$ contains all the functions which are analytic, univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. That is

$$\mathcal{S} = \{f : f \in \mathcal{A},\ f(\mathrm{z}) \text{ is univalent in } \mathbb{E}\}.$$

The well known example of the class $\mathcal{S}$ is the Koebe function

$$k(\mathrm{z}) = \frac{\mathrm{z}}{(1-\mathrm{z})^2} = \mathrm{z} + \sum_{\ell=2}^{\infty} \ell \mathrm{z}^{\ell}, \qquad \mathrm{z} \in \mathbb{E}. \tag{2.1.2}$$

The function k plays an imperative role in the class $\mathcal{S}$ because of its extremal nature. The Koebe function maps $\mathbb{E}$ onto the entire complex plane minus negative real axis from $-\frac{1}{4}$ to $-\infty$. The class $\mathcal{S}$ is preserved under a number of elementary transformations, see [17, 24] such as dilation, conjugation, rotation and disk automorphism. In 1985, de Branges settled the well-known problem in the univalent function theory, by proving the Bieberbach conjecture for the coefficient estimates of the class $\mathcal{S}$ is that $|a_\ell| \leq \ell$ holds for $\ell \geq 2$.

Here is the Bieberbach's theorem which states as follows.

Lemma 2.1.1. [15] Let

$$f(\mathrm{z}) = \mathrm{z} + \sum_{\ell=2}^{\infty} a_\ell \mathrm{z}^{\ell} \in \mathcal{S}.$$

Then $|a_2| \leq 2$ and this inequality is sharp. Equality holds for some rotation of Koebe function given by (2.1.2).

In the following, we give distortion bounds, for functions in the class $\mathcal{S}$. For reference, see [17, 24].

Lemma 2.1.2. Let $f \in \mathcal{S}$ and let $\mathrm{z} = \mathrm{r}e^{i\theta} \in \mathbb{E}$, then

$$\frac{\mathrm{r}}{(1+\mathrm{r})^2} \leq |f(\mathrm{z})| \leq \frac{\mathrm{r}}{(1-\mathrm{r})^2},$$

and

$$\frac{1-\mathrm{r}}{(1+\mathrm{r})^3} \leq \left|f'(\mathrm{z})\right| \leq \frac{1+\mathrm{r}}{(1-\mathrm{r})^3}.$$

These results are sharp. Koebe function and some of its rotations provide equality in the above relations.

2.2 The Class $\mathcal{P}$ and Related Classes

There are several complex valued functions in Geometric Function Theory whose image domains cover the entire complex plane, there also exist functions with image domains restricted to the right half plane. Such functions were required to be normalized as it was ended in the study of univalent functions. We named $\mathcal{P}$, to be the class of said functions, see [24]. Some related classes will also be introduced, however, some of their basic properties are given.

Definition 2.2.1. [17, 24] The analytic function h which satisfy the conditions $h(0) = 1$ and $\mathfrak{Re}\{h(\mathrm{z})\} > 0$, $\mathrm{z} \in \mathbb{E}$ are said to form the class $\mathcal{P}$. That is

$$\left\{h \in \mathcal{P} : h(\mathrm{z}) = 1+\sum_{\ell=1}^{\infty} d_\ell \mathrm{z}^\ell, \quad \text{iff} \quad \mathfrak{Re}\{h(\mathrm{z})\} > 0, \quad \mathrm{z} \in \mathbb{E}\right\}.$$

The leading example of the class $\mathcal{P}$ is the Möbius function

$$L_o(\mathrm{z}) = \frac{1+\mathrm{z}}{1-\mathrm{z}} = 1+2\sum_{\ell=1}^{\infty} \mathrm{z}^\ell, \quad \mathrm{z} \in \mathbb{E}, \tag{2.2.1}$$

which plays the part of extremal function for this class in many cases.

We observe that

i) The functions belonging to the class $\mathcal{P}$ need not be univalent. The function $h(\mathrm{z}) = 1+\mathrm{z}^\ell$ is in the class $\mathcal{P}$ for $\ell \geq 0$, but for $\ell \geq 2$ it is not univalent.

ii) Let $h \in \mathcal{P}$. Then, for $\ell \geq 1$, $|d_\ell| \leq 2$. This inequality is sharp for L_0, given in (2.2.1).

iii) The set $\mathcal{P}$ is convex. This mean that if α_1 and α_2 are non-negative with $\alpha_1+\alpha_2 = 1$ and $h_1,\ h_2$ are in $\mathcal{P}$, then $h\left(\mathrm{z}\right) = \alpha_1 h_1\left(\mathrm{z}\right) + \alpha_2 h_2\left(\mathrm{z}\right)$, is also in $\mathcal{P}$, see [17, 24].

In 1911, Herglotz [26] defined the function from class $\mathcal{P}$ in different technique by introducing their integral representation as:

$$h(\mathrm{z}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\mathrm{z}e^{-it}}{1-\mathrm{z}e^{-it}} d\mu(t), \qquad \text{for all } \mathrm{z} \in \mathbb{E},$$

where $\mu(t)$ is a non-decreasing real valued function such that

$$\int_0^{2\pi} d\mu(t) = 2\pi.$$

Noshiro and Warschawski [84, 127] gave sufficient condition for univalency in 1935, as follows.

Theorem 2.2.1. [84, 127] Let θ be real and for all z in a convex domain D. If

$$\mathfrak{Re}\left\{e^{i\theta} f'\left(\mathrm{z}\right)\right\} \geq 0,$$

then $f\left(\mathrm{z}\right)$ is univalent in $\mathbb{E}$.

Lemma 2.2.1. Let $h \in \mathcal{P}$ be of the form

$$h\left(\mathrm{z}\right) = 1+\sum_{\ell=1}^{\infty} d_\ell \mathrm{z}^\ell, \quad \text{then} \quad |d_\ell| \leq 2,\ \ell = 1, 2, \ldots, \tag{2.2.2}$$

equality holds for some rotation of the Möbius function given by (2.2.1).

Lemma 2.2.2. [17, 24] Let $h \in \mathcal{P}$. Then for $|\mathrm{z}| = \mathrm{r} < 1$

$$\frac{1-\ \mathrm{r}}{1+\ \mathrm{r}} \leq \mathfrak{Re} h(\mathrm{z}) \leq |h(\mathrm{z})| \leq \frac{1+\ \mathrm{r}}{1-\ \mathrm{r}}, \tag{2.2.3}$$

$$|\mathrm{z}h'(\mathrm{z})| \leq \frac{2\mathrm{r}\ \mathfrak{Re} h(\mathrm{z})}{1-\ \mathrm{r}^2}. \tag{2.2.4}$$

The bounds (2.2.3) and (2.2.4) are sharp and equalities are attained, iff, h is a suitable rotation of the Möbius function defined by (2.2.1), see [17, 24].

Definition 2.2.3. Let the collection $\mathcal{P}(\lambda)$ containing the functions h with $\mathfrak{Re}\{h(\mathrm{z})\} > \lambda$ in $\mathbb{E}$ where, $0 \leq \lambda < 1$. If $h \in \mathcal{P}(\lambda)$, then we can write $h(\mathrm{z})$ as follows.

$$h(\mathrm{z}) = (1-\lambda) h_1(\mathrm{z}) + \lambda, \quad h_1 \in \mathcal{P}, \quad \mathrm{z} \in \mathbb{E}.$$

We note that $\mathcal{P}(0) = \mathcal{P}$.

Lemma 2.2.3. [17, 24] Let $h \in \mathcal{P}(\lambda)$ and be of the form (2.2.2). Then

$$|d_\ell| \leq 2(1-\lambda), \text{ for all } \ell \geq 1.$$

Lemma 2.2.4. Let $h \in \mathcal{P}(\lambda)$. Then for $|\mathrm{z}| = \mathrm{r} < 1$,

$$\frac{1-\ (1-2\lambda)\mathrm{r}}{1+\ \mathrm{r}} \leq |h(\mathrm{z})| \leq \frac{1+\ (1-2\lambda)\mathrm{r}}{1-\ \mathrm{r}}, \ \mathrm{z} = \mathrm{r}e^{i\theta}.$$

For detail see [17, 24].

2.3 Certain Subclasses of Univalent Functions

We will consider some subclasses of normalized univalent functions in this section which are defined on the basis of geometry of their image domains. Some important properties of these classes will be discussed. Also their relationship with the class $\mathcal{P}$ is discussed.

Definition 2.3.1. [17, 24] A set $\mathfrak{D} \subset \mathbb{C}$ is called starlike with respect to $\mathrm{z}_0 \in \mathfrak{D}$ if each ray with initial point z_0 intersects the interior of $\mathfrak{D}$ in a set that is either a line segment or a ray. A function f which maps $\mathbb{E}$ onto a starshaped domain with respect to the point z_0 is said to be starlike function. The function defined in (2.1.2) is the well known example of starlike function.

Definition 2.3.2. [17, 24] A set $\mathfrak{D} \subset \mathbb{C}$ is named as convex if for all pair of points $z_1, z_2 \in \mathfrak{D}$, the line segment joining z_1 and z_2 is also in the interior of $\mathfrak{D}$. A function f which maps $\mathbb{E}$ onto a convex domain is said to be convex function. The prime example of the convex function is given as:

$$f(z) = \frac{z}{1-z}. \tag{2.3.1}$$

Theorem 2.3.1. [62] Let $f \in \mathcal{S}$. Then $f(z)$ maps $\mathbb{E}$ onto a starshaped domain, iff

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{E}. \tag{2.3.2}$$

or

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}.$$

For example Koebe function given in (2.1.2) is a starlike function which plays the role of extremel function for this class. It is noted that

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}.$$

The following necessary and sufficient condition for convex functions is due to Study [121].

Theorem 2.3.2. [121] Let $f \in \mathcal{S}$. Then f maps $\mathbb{E}$ onto a convex domain, iff

$$\mathfrak{Re}\frac{(zf'(z))'}{f'(z)} > 0, \quad z \in \mathbb{E},$$

or equivalently

$$\frac{(zf'(z))'}{f'(z)} \in \mathcal{P}, \quad z \in \mathbb{E}.$$

In 1915, Alexander [2] proved the connection between starlike and convex functions as follows.

$$f \in \mathcal{C} \Longleftrightarrow \quad \mathrm{z}f' \in \mathcal{S}^*.$$

Lemma 2.3.1. [17, 24] Let $f \in \mathcal{S}^*$ and be given by (2.1.1). Then for $\mathrm{z} \in \mathbb{E}$,

$$|a_\ell| \leq \ell, \quad \ell \geq 2.$$

Equality holds for a suitable rotation of the Koebe function given by (2.1.2).

Lemma 2.3.2. [17, 24] Let $f \in \mathcal{S}^*$ and be given by (2.1.1). Then for each $|\mathrm{z}| = \mathrm{r} < 1$,

$$\frac{\mathrm{r}}{(1+\mathrm{r})^2} \leq |f(\mathrm{z})| \leq \frac{\mathrm{r}}{(1-\mathrm{r})^2},$$

and

$$\frac{1-\mathrm{r}}{(1+\mathrm{r})^3} \leq |f'(\mathrm{z})| \leq \frac{1+\mathrm{r}}{(1-\mathrm{r})^3}.$$

These inequalities are sharp and equality occurs for some rotation of the function given in (2.1.2).

Lemma 2.3.3. [17, 24] Let $f \in \mathcal{C}$ and be given by (2.1.1). Then for $\mathrm{z} \in \mathbb{E}$,

$$|a_\ell| \leq 1, \quad \text{for all} \quad \ell \geq 2, \tag{2.3.3}$$

The bound given in (2.3.3) is sharp and equality holds for the function given in (2.3.1).

Lemma 2.3.4. [17, 24] Let $f \in \mathcal{C}$. Then for each $|\mathrm{z}| = \mathrm{r} < 1$,

$$\frac{\mathrm{r}}{1+\mathrm{r}} \leq |f(\mathrm{z})| \leq \frac{\mathrm{r}}{1-\mathrm{r}},$$

and

$$\frac{1}{(1+\mathrm{r})^2} \leq |f'(\mathrm{z})| \leq \frac{1}{(1-\mathrm{r})^2}.$$

These results are sharp and equality occurs for some rotation of the function given by (2.1.2).

In 1936, Robertson [102] state the followings.

Definition 2.3.3. [17, 24] A function $f \in \mathcal{S}^*(\lambda)$, $0 \leq \lambda < 1$, iff

$$\mathfrak{Re}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} > \lambda, \quad \mathrm{z} \in \mathbb{E}.$$

Definition 2.3.4. A function $f \in \mathcal{C}(\lambda)$, $0 \leq \lambda < 1$, iff

$$\mathfrak{Re}\frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})} > \lambda, \quad \mathrm{z} \in \mathbb{E}.$$

For detail see [17, 24].

In 1952, Kaplan [37] investigated the class $\mathcal{K}$ which contains the class $\mathcal{S}^*$ of starlike functions. He defined the class $\mathcal{K}$ as follows.

Definition 2.3.5. Let $f \in \mathcal{A}$. Then $f \in \mathcal{K}$ if there exists a starlike function g in such a way that

$$\mathfrak{Re}\frac{\mathrm{z}f'(\mathrm{z})}{g(\mathrm{z})} > 0, \quad \mathrm{z} \in \mathbb{E}.$$

Using Alexander relation the above inequality can also be written as:

$$\mathfrak{Re}\frac{f'(\mathrm{z})}{F'(\mathrm{z})} > 0, \quad \mathrm{z} \in \mathbb{E}.$$

for some $F \in \mathcal{C}$. It was proved in [37] that close-to-convex functions are univalent. Every convex function is close-to-convex function. More generally, every starlike function is close-to-convex. It is noted that

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K}.$$

In 1969, Mocanu [61] gave the idea of $\alpha-$convexity as follows.

Definition 2.3.6. [61] Let $f \in \mathcal{A}$ and $\frac{f(z)f'(\mathrm{z})}{\mathrm{z}} \neq 0$. Then $f \in \mathcal{M}(\alpha)$ for $\alpha \in \mathbb{R}$, iff

$$\mathfrak{Re}\left\{(1-\alpha)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} + \alpha\frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})}\right\} > 0, \text{ for all } \mathrm{z} \in \mathbb{E}.$$

The above class is also meaningful if we take λ to be a complex number, but we assume

α to be real. We see that

$$\mathcal{M}(0) \equiv \mathcal{S}^* \text{ and } \mathcal{M}(1) = \mathcal{C}.$$

Miller, Mocanu and Read [58] proved that $\mathcal{M}(\alpha) \subset \mathcal{S}^*$ for $\alpha \geq 0$ and $\mathcal{M}(\alpha) \subset \mathcal{C}$ for $\alpha \geq 1$.

2.3.1 Spirallike functions of type β and Robertson Functions [27, 39]

Let β be a real number between $\frac{\pi}{2}$ and -$\frac{\pi}{2}$. The curve

$$\gamma_\beta : t \to \exp(te^{i\beta}), \quad t \in \mathbb{R},$$

and their rotations $e^{i\theta}\gamma_\beta$, $\theta \in \mathbb{R}$, are called $\beta-$spirals. These curves γ_β are characterized by the property that the oriented angle from γ_β to the tangent vector γ_β', which is called the radial angle, is constantly β that is

$$\arg\left(\frac{\gamma_\beta'}{\gamma_\beta}\right) = \beta.$$

Also, we see that this curve family is invariant under the dilation $\mathrm{z} \to \mathrm{cz}$ for $\mathrm{c} \in \mathbb{C}\backslash\{0\}$. For $\mathrm{z}_0 \in \mathbb{C}$, we define the β-spiral segment $[0, \mathrm{z}_0]_\beta$ by

$$[0, \mathrm{z}_0]_\beta = \mathrm{z}_0.\gamma_\beta(-\infty, 0) \cup \{0\} = \left\{\mathrm{z}_0 \exp(te^{i\beta}) : t \leq 0\right\} \cup \{0\}.$$

Clearly, $[0, \mathrm{z}_0]_0$ is the line segment $[0, \mathrm{z}_0]$.

A domain $\mathfrak{D}$ with $0 \in \mathfrak{D}$ is called $\beta-$spirallike with respect to origin if $[0, \mathrm{z}_0]_\beta \subset \mathfrak{D}$ whenever $\mathrm{z}_0 \in \mathfrak{D}$.

Definition 2.3.7. A function f which maps $\mathbb{E}$ onto a domain that is $\beta-$spirallike with respect to the origin is called spirallike function of type β, we symbolize by $\mathcal{S}_\beta$ to be the class of said functions.

The following analytic condition for spirallike functions is due to Specek [118].

Theorem 2.3.3. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_\beta$, $|\beta| < \frac{\pi}{2}$ of

$\beta-$spirallike functions, iff

$$\mathfrak{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbb{E}.$$

Specek also showed that these functions are univalent, that is, $\mathcal{S}_\beta \subset \mathcal{S}$, see [118, 24].

In 1967, Libera [44] extended this definition to functions spirallike of order λ denoted by $\mathcal{S}_\beta(\lambda)$ as follows.

Theorem 2.3.4. Let $f \in \mathcal{A}$. Then f is called $\beta-$spirallike of order λ $(0 \leq \lambda < 1)$, iff

$$\mathfrak{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \lambda\cos\beta, \qquad z \in \mathbb{E}.$$

Lemma 2.3.5. Let f be a $\beta-$spirallike function. Then, for $z = re^{i\theta}$

$$\frac{r}{(1+r)^{2\cos\beta}} \leq |f(z)| \leq \frac{r}{(1-r)^{2\cos\beta}}.$$

For more detail, we refer to [94].

In 1972, Libera and Zeigler [45] derived the following relation between the class of starlike and the class of spirallike functions.

Lemma 2.3.6. Let $f \in \mathcal{A}$ and $|\beta| < \frac{\pi}{2}$. Then $f \in \mathcal{S}_\beta$, iff

$$f(z) = z\left(\frac{f_1(z)}{z}\right)^{\cos\beta e^{-i\beta}},$$

for some $f_1 \in \mathcal{S}^*$.

Robertson functions

In 1969, Robertson considered the class $\mathcal{C}_\beta$ of analytic functions defined as:

Definition 2.3.8. A function $f \in \mathcal{A}$ is called Robertson if $zf' \in \mathcal{S}_\beta$.

Robertson proved that $f \in \mathcal{C}_\beta$ is univalent in $\mathbb{E}$ if $\frac{1}{2} < \cos\beta < 1$. Later Libera and Zeigler [45] and Chichra [8] improved upon the range of $\cos\beta$ for which $f \in \mathcal{C}_\beta$ is univalent in $\mathbb{E}$.

Theorem 2.3.5. Let $f \in \mathcal{A}$. Then $f \in \mathcal{C}_\beta$, $|\beta| < \frac{\pi}{2}$ of Robertson functions, iff

$$\mathfrak{Re}\left[e^{i\beta}\left(1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right] > 0, \qquad \mathrm{z} \in \mathbb{E}.$$

Lemma 2.3.7. [8] Let $f \in \mathcal{A}$ and $|\beta| < \frac{\pi}{2}$. Then $f \in \mathcal{C}_\beta$, iff

$$f'(\mathrm{z}) = \left(\frac{f_1(\mathrm{z})}{\mathrm{z}}\right)^{\cos\beta e^{-i\beta}},$$

for some $f_1 \in \mathcal{S}^*$.

Definition 2.3.9. Let $f \in \mathcal{A}$ and $|\beta| < \frac{\pi}{2}$. Then $f \in \mathcal{SC}_\beta$, $\beta-$spirallike convex function, iff

$$\mathfrak{Re}\left\{\cos\left(1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)+i\sin\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right\} > 0, \quad \mathrm{z} \in \mathbb{E} \tag{2.3.4}$$

Yoshikawa [128] introduced the class $\mathcal{SC}_\beta$ and showed that $\mathcal{SC}_\beta \subset \mathcal{S}_\beta$.

In 1974, Silvia [116] defined a class which contains the classes $\mathcal{M}(\alpha)$ and $\mathcal{S}_\beta(\lambda)$ as special cases.

Definition 2.3.10. Let $f \in \mathcal{A}$ with $f(\mathrm{z})f'(\mathrm{z}) \neq 0$ in $\mathbb{E}$. Then $f \in \mathcal{M}(\alpha,\beta,\lambda)$, the class of $\alpha-\beta-$spirallike of order λ if

$$\mathfrak{Re}\left\{\left(e^{i\beta}-\alpha\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\alpha\left(1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right\} > \lambda\cos\beta, \quad \mathrm{z} \in \mathbb{E},$$

where $\alpha \geq 0$, $|\beta| < \frac{\pi}{2}$ and $0 \leq \lambda < 1$.

We note that $\mathcal{M}(0,\beta,\lambda) = \mathcal{S}_\beta(\lambda)$ and $\mathcal{M}(\alpha,0,0) = \mathcal{M}(\alpha)$.

2.4 Multivalent Functions

In the present section we shall discuss the class $\mathcal{A}(p)$ of analytic multivalent functions(p-valent functions) in $\mathbb{E}$. Also we give the definitions of certain subclasses of multivalent functions such as starlike and convex and their properties. The class of multivalent functions plays a significant role in Complex Analysis, see for example [17, 24, 27].

Definition 2.4.1. A function $f \in \mathcal{A}(p)$, the class of analytic multivalent functions if it has the power series representation

$$f(\mathrm{z}) = \mathrm{z}^p + \sum_{\ell=1}^{\infty} a_{p+\ell}\mathrm{z}^{p+\ell}, \qquad p \in \mathbb{N}. \tag{2.4.1}$$

We note that $\mathcal{A}(1) = \mathcal{A}$ as given by (2.1.1).

Definition 2.4.2. [27] Let $f \in \mathcal{A}(p)$ and given by (2.4.1). Then f is multivalent starlike in $\mathbb{E}$, iff

$$\mathfrak{Re}\left(\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right) > 0, \qquad (\mathrm{z} \in \mathbb{E}).$$

We named as $\mathcal{S}_p^*$ to be the class of said functions. For $p = 1$, $\mathcal{S}_1^* = \mathcal{S}^*$, the well known class of starlike functions.

Definition 2.4.3. [27] Let $f \in \mathcal{A}(p)$ and given by (2.4.1). Then f is multivalent convex in $\mathbb{E}$, iff

$$\mathfrak{Re}\left(1 + \frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right) > 0, \qquad (\mathrm{z} \in \mathbb{E}).$$

The class contains all multivalent convex functions is denoted by $\mathcal{C}_p$. We note that $\mathcal{C}_1 = \mathcal{C}$. The Alexader type relation also holds for these classes, that is

$$f \in \mathcal{C}_p \quad \text{iff} \quad \frac{\mathrm{z}f'}{p} \in \mathcal{S}_p^*.$$

In 1985, Owa [90] introduced the concept of order of analytic multivalent functions and further it was studied by Srivastava and Owa [120].

Definition 2.4.4. [27] Let $f \in \mathcal{A}(p)$. Then $f \in \mathcal{S}_p^*(\lambda)$ the class of multivalent starlike functions of order λ $(0 \leq \lambda < p)$ in $\mathbb{E}$, iff

$$\mathfrak{Re}\left(\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right) > \lambda, \qquad (\mathrm{z} \in \mathbb{E}).$$

It is easy to see that $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$ and $\mathcal{S}_1^* = \mathcal{S}^*$.

Definition 2.4.5. [27] Let $f \in \mathcal{A}(p)$. Then $f \in \mathcal{C}_p(\lambda)$, the class of multivalent convex functions of order λ $(0 \leq \lambda < p)$, iff

$$\mathfrak{Re}\left(1+ \frac{zf''(z)}{f'(z)}\right) > \lambda, \qquad (z \in \mathbb{E}).$$

It is easy to be see that $\mathcal{C}_p(0) = \mathcal{C}_p$ and $\mathcal{C}_1 = \mathcal{C}$.

2.5 Subordination and Differential Subordination

Lindelöf [47] introduced the idea of subordination which was further studied by Littlewood [49] and Rogosinski [105]. The idea of subordination is based on the Schwarz functions. So the definition of Schwarz function is provided first and after that we discuss the definition of subordination.

Definition 2.5.1. Let Λ be the collection of all analytic functions w in $\mathbb{E}$, satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{E}$. Then the function $w(z)$ is said to be a Schwarz function.

Definition 2.5.2. If the functions f and g both are analytic in $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$, written as $f \prec g$, iff there exists a Schwarz function w in such a way that

$$f(z) = g(w(z)), \qquad z \in \mathbb{E}.$$

Therefore, $f \prec g$ in $\mathbb{E}$ implies $f(\mathbb{E}) \subset g(\mathbb{E})$. If g is univalent in $\mathbb{E}$, then the Subordination Principle says that $f \prec g$, iff $f(0) = g(0)$ and $f(|z| < r) \subset g(|z| < r)$, for all $r \in (0, 1)$.

Differential Subordination

In 1981, Miller and Mocanu [56] gave the concept of differential subordination. In fact, a differential subordination on the complex plane is a generalization of differential inequalities on the real line. The first order differential subordination has many applications in the theory of univalent functions. Differential subordination has developed itself as a field and has many application, such as in the area of differential inequalities, integral operators and many others.

Definition 2.5.3. Let $\psi : \mathbb{C}^3 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If g is analytic in $\mathbb{E}$ and satisfy the differential subordination

$$\psi\left(g(\mathrm{z}), \mathrm{z}g'(\mathrm{z}), \mathrm{z}^2 g''(\mathrm{z}); \mathrm{z}\right) \prec h, \tag{2.5.1}$$

then g is called the solution of differential subordination. The univalent function $q(\mathrm{z})$ is called a dominant of the differential subordination (2.5.1) if $g \prec q$ for all g satisfying (2.5.1). If $\tilde{q}(\mathrm{z})$ is a dominant of (2.5.1) and $\tilde{q} \prec q$ for all dominants of (2.5.1), then $\tilde{q}$ is said to be the best dominant of (2.5.1).

2.6 Conic Type Regions

In this section, we will discuss conic domains, circular domains and some associated functions. The classes of $k-$uniformly convex (starlike) functions were studied in [22, 23, 33, 34] where their geometric definitions and relations with conic domains were established.

2.6.1 Uniformly Convex (Starlike) Functions

The class of uniformly convex (starlike) functions was first investigated in [22, 23], and further studied by Rønning [107], Ma and Minda [53], and others. The analytic conditions for these classes are given as follows.

Definition 2.6.1. [22, 23] Let $f \in \mathcal{A}$, and be given by (2.1.1). Then $f \in \mathcal{UCV}$, the class of uniformly convex functions, iff

$$\mathfrak{Re}\left[1 + (\mathrm{z}-\xi)\frac{f''(\mathrm{z})}{f'(\mathrm{z})}\right] > 0, \quad (\mathrm{z}, \xi) \in \mathbb{E} \times \mathbb{E}.$$

Definition 2.6.2. [22, 23] Let $f \in \mathcal{A}$, and be given by (2.1.1). Then $f \in \mathcal{ST}$, the class of uniformly starlike functions, iff

$$\mathfrak{Re}\left[\frac{(\mathrm{z}-\xi)f'(\mathrm{z})}{f(\mathrm{z}) - f(\xi)}\right] > 0, \quad (\mathrm{z}, \xi) \in \mathbb{E} \times \mathbb{E}.$$

By taking $\xi = 0$ in above definitions we are back to the classes $\mathcal{C}$ and $\mathcal{S}^*$. The famous Alexander's theorem gives a connection between these $(\mathcal{C}, \mathcal{S}^*)$ two classes. One might hope that there would be a similar connection between $\mathcal{UCV}$ and $\mathcal{ST}$, but two examples in [22] showed that this is not the case. In 1994, Ma and Minda [53] gave an analytical and more appropriate, one variable characterizations, of the class $\mathcal{UCV}$ and the related class $\mathcal{ST}$, as follows.

Definition 2.6.3. [53] Let $f \in \mathcal{A}$, and be given by (2.1.1). Then f is in the class $\mathcal{UCV}$, iff

$$\Re\mathfrak{e}\left[1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right] > \left|\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right|, \qquad \mathrm{z} \in \mathbb{E}.$$

Definition 2.6.4. [53] Let $f \in \mathcal{A}$, and be given by (2.1.1). Then f is in the class $\mathcal{ST}$, iff

$$\Re\mathfrak{e}\left[\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right] > \left|\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}-1\right|, \qquad \mathrm{z} \in \mathbb{E}.$$

This analytical one variable characterization provided the first conic (parabolic) domain

$$\Omega = \{w : \Re\mathfrak{e}w(\mathrm{z}) > |w(\mathrm{z})-1|\}.$$

Kanas and Wisniowska [33, 34] generalized the above parabolic domain and established the conic domain Ω_k, $k \geq 0$ and studied it in detail. Let $k \in [0,\infty)$. For arbitrary chosen k, let Ω_k denote the following domain

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}.$$

We see that Ω_k is convex and symmetric in the real axis and $1 \in \Omega_k$ for all k. For $k = 0$, Ω_0 is nothing but the right half-plane and when $0 < k < 1$, Ω_k represent hyperbolic regions (right branch), a parabolic region when $k = 1$ and elliptic region when $k > 1$. It should be noted that, for no choice of parameter k, Ω_k reduces to a disk. The boundaries of conic regions are shown in figures below. For $k = \frac{1}{4}$, $k = \frac{1}{100}$ and for $k = 2$, $k = 3$ we obtain figure 2.6.2 and figure 2.6.3 respectively.

The functions which play the role of extremal functions for these conic regions are

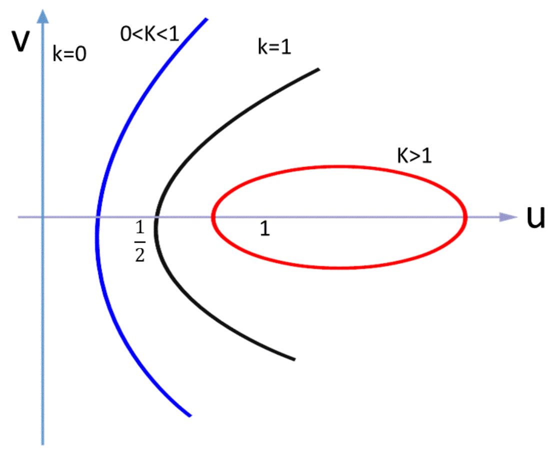

Figure 2.6.1. Conic regions

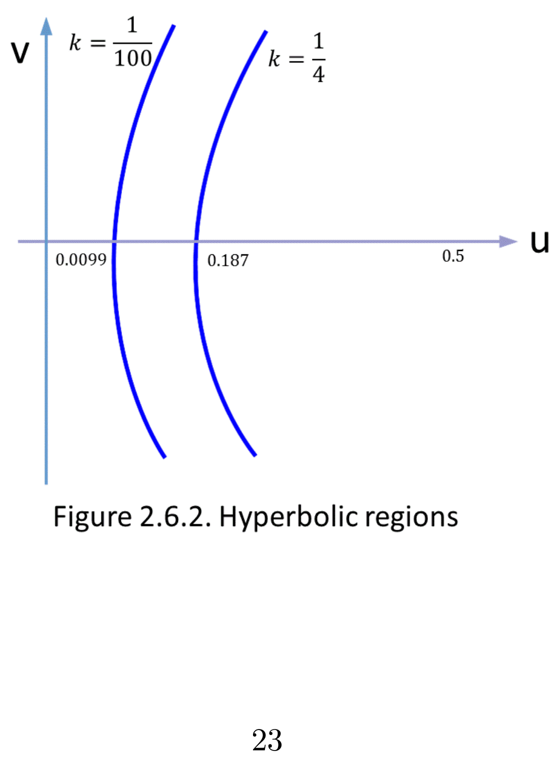

Figure 2.6.2. Hyperbolic regions

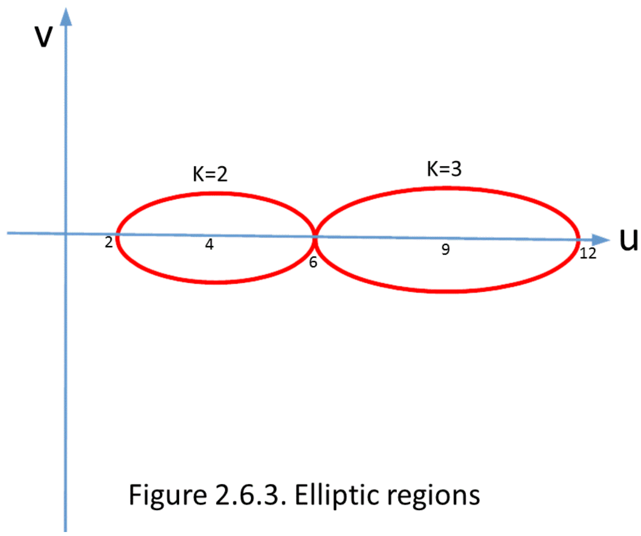

Figure 2.6.3. Elliptic regions

given as:

$$q_k\left(z\right)=\begin{cases}\frac{1+z}{1-z}, & k=0,\\ 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1,\\ 1+\frac{2}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\}, & 0<k<1,\\ 1+\frac{2}{k^2-1}\sin\left(\frac{\pi}{2K(t)}\int_0^{\frac{\mu(z)}{t}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(xt)^2}}dx\right)+\frac{1}{k^2-1}, & k>1,\end{cases} \tag{2.6.1}$$

where $\mu(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t\in(0,1)$ and z is chosen such that $k=\cosh\left(\pi K'(t)/(4K(t))\right)$, $K(t)$ is the Legendre's complete elliptic integral of the first kind and $K'(t)$ is complementary integral of $K(t)$, see [33, 34].

For $k\in[0,\infty)$, Ω_k forms the family of domains bounded by conic sections. For comprehensive study of these conic regions, see [74, 75, 114]. The class $\mathcal{P}(q_k)$ of functions which map $\mathbb{E}$ onto these conic regions is defined in [33] as follows.

Definition 2.6.5. A function h with $h(0)=1$ is in the class $\mathcal{P}(q_k)$ if it subordinate to $q_k(z)$, $z\in\mathbb{E}$.

It is shown in [33] that

i) $h(\mathbb{E})\subset q_k(\mathbb{E})=\Omega_k$.

ii) $\mathfrak{Re}h(\mathrm{z}) > \frac{k}{k+1}$, that is, $\mathcal{P}(q_k) \subset \mathcal{P}\left(\frac{k}{k+1}\right)$.

In 1999, 2000, the classes $k-\mathcal{UCV}$, of $k-$uniformly convex functions and $k-\mathcal{ST}$, of $k-$uniformly starlike functions were studied and their relation with the conic domains were measured [33, 34]..

By making use of the class $\mathcal{P}(q_k)$, we can state the followings.

Definition 2.6.6. [33, 34] Let $f \in \mathcal{A}$. Then f is in the class $k-\mathcal{UCV}$, iff

$$\mathfrak{Re}\left[\frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})}\right] > k\left|\frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})}-1\right|, \qquad k \geq 0, \quad \mathrm{z} \in \mathbb{E},$$

or equivalently

$$\frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})} \prec q_k(\mathrm{z}), \qquad k \geq 0, \quad \mathrm{z} \in \mathbb{E},$$

where $q_k(\mathrm{z})$ are given by (2.6.1).

Definition 2.6.7. [33, 34] Let $f \in \mathcal{A}$. Then f is in the class $k-\mathcal{ST}$, iff

$$\mathfrak{Re}\left[\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right] > k\left|\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}-1\right|, \qquad k \geq 0, \quad \mathrm{z} \in \mathbb{E},$$

or equivalently

$$\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} \prec q_k(\mathrm{z}), \qquad k \geq 0, \quad \mathrm{z} \in \mathbb{E}.$$

where $q_k(\mathrm{z})$ are given by (2.6.1).

Theorem 2.6.1. Let $f \in \mathcal{S}$. Then $f \in k-\mathcal{UCV}$ for $|\mathrm{z}| < \mathrm{r}_0$, where

$$\mathrm{r}_0 = \frac{1}{2(k+1)\sqrt{4k^2+6k+3}}.$$

For this result, we refer to [33].

Circular Domains

In 1973, Janowski [30] introduced and studied the circular domain by defining Janowski functions. These functions are defined as:

Definition 2.6.8. Let h be analytic with $h(0) = 1$. Then $h \in \mathcal{P}[\mathbb{A}, \mathbb{B}]$ if

$$h(\mathrm{z}) \prec \frac{1+\mathbb{A}\mathrm{z}}{1+\mathbb{B}\mathrm{z}}, \quad -1 \leq \mathbb{B} < \mathbb{A} \leq 1.$$

Geometrically, a function $h \in \mathcal{P}[\mathbb{A}, \mathbb{B}]$ maps $\mathbb{E}$ onto the domain $\Omega[\mathbb{A}, \mathbb{B}]$ defined by

$$\Omega[\mathbb{A}, \mathbb{B}] = \left\{ w : \left| w - \frac{1-\mathbb{A}\mathbb{B}}{1-\mathbb{B}^2} \right| < \frac{\mathbb{A}-\mathbb{B}}{1-\mathbb{B}^2} \right\}.$$

The classes $\mathcal{P}[\mathbb{A}, \mathbb{B}]$ and $\mathcal{P}$ are connected each other by the relation given as:

$$h(\mathrm{z}) \in \mathcal{P} \quad \text{iff} \quad \frac{(\mathbb{A}+1)\, h(\mathrm{z}) - (\mathbb{A}-1)}{(\mathbb{B}+1)\, h(\mathrm{z}) - (\mathbb{B}-1)} \in \mathcal{P}[\mathbb{A}, \mathbb{B}].$$

We observe that $\mathcal{P}[1, -1] = \mathcal{P}$. Also it is known [76] that $\mathcal{P}[\mathbb{A}, \mathbb{B}]$ is a convex set.
The classes $\mathcal{C}[\mathbb{A}, \mathbb{B}]$ and $\mathcal{S}^*[\mathbb{A}, \mathbb{B}]$ of Janowski convex and Janowski starlike functions were also defined by Janowski [30] as follows.

Definition 2.6.9. A function $f \in \mathcal{A}$ is in the class $\mathcal{C}[\mathbb{A}, \mathbb{B}]$, iff

$$\frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})} \in \mathcal{P}[\mathbb{A}, \mathbb{B}].$$

Definition 2.6.10. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*[\mathbb{A}, \mathbb{B}]$, iff

$$\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} \in \mathcal{P}[\mathbb{A}, \mathbb{B}].$$

The Alexander type relation also holds for these classes, that is

$$f \in \mathcal{C}[\mathbb{A}, \mathbb{B}] \qquad \text{iff} \qquad \mathrm{z}f' \in \mathcal{S}^*[\mathbb{A}, \mathbb{B}].$$

These classes are further investigated by several well-known mathematicians, see, for example, [10, 11, 50, 51, 77, 78, 95].

2.7 The Class of Bounded Boundary and Bounded Radius Rotation

In this section we give the definition of the class of bounded boundary rotation which was defined by Lowner [52] and further investigated by Paatero [92, 93], and others. The corresponding class, class of bounded radius rotations was defined by Tammi [122] in 1982.

Definition 2.7.1. Let $f \in \mathcal{A}$. Then $f \in \mathcal{V}_m$ the class of bounded boundary rotation, if the variation of tangent angle at the boundary of $f(\mathbb{E})$ is at most $m\pi$, $m \geq 2$. That is

$$\int_0^{2\pi} \left| \mathfrak{Re} \left(\frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})} \right) \right| d\theta \leq m\pi, \qquad m \geq 2, \quad \mathrm{z} \in \mathbb{E}.$$

We note that $\mathcal{V}_2 = \mathcal{C}$. Paatero [93] showed that for $m \in [2, 4]$ the functions from $\mathcal{V}_m$ are univalent. However, for $m > 4$ the functions from class $\mathcal{V}_m$ are non-univalent, and radius of univalency is $r = \tan \frac{\pi}{m}$, see [42].

Definition 2.7.2 Let $f \in \mathcal{A}$. Then f is in the class $\mathcal{R}_m$ of bounded radius rotation if

$$\int_0^{2\pi} \left| \mathfrak{Re} \left(\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} \right) \right| d\theta \leq m\pi, \qquad m \geq 2, \quad \mathrm{z} \in \mathbb{E}.$$

It is noted that $\mathcal{R}_2 = \mathcal{S}^*$, the class of starlike univalent functions. The famous Alexander's type relation holds between $\mathcal{V}_m$ and $\mathcal{R}_m$, that is

$$f \in \mathcal{V}_m \qquad \text{iff} \qquad \mathrm{z}f' \in \mathcal{R}_m$$

The Class $\mathbf{P}_m$

In 1971, Pinchuk [97] introduced the class $\mathcal{P}_m$ of functions with bounded turnings. These functions are defined as follows.

Definition 2.7.3. A function h with $h(0) = 1$ is in the class $\mathcal{P}_m$, if it is analytic in $\mathbb{E}$ and

$$h(\mathrm{z}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1+\mathrm{z}e^{-it}}{1-\mathrm{z}e^{-it}} d\mu(t), \qquad \mathrm{z} \in \mathbb{E},$$

where $\mu(t)$ is a non-decreasing real valued function with bounded variation on $[0, 2\pi]$ such that for $m \geq 2$,

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m.$$

The classes $\mathcal{P}_m$ and $\mathcal{P}$ are connected each other by the relation given as:

For $h \in \mathcal{P}_m$, we have

$$h(\mathrm{z}) = \left(\frac{m}{4}+\frac{1}{2}\right) h_1(\mathrm{z}) - \left(\frac{m}{4}-\frac{1}{2}\right) h_2(\mathrm{z}), \tag{2.7.1}$$

where $h_1(\mathrm{z}), h_2(\mathrm{z}) \in \mathcal{P}$. It is noted that $\mathcal{P}_2 = \mathcal{P}$.

The classes $\mathcal{V}_m$ and $\mathcal{R}_m$ are also defined by using the class $\mathcal{P}_m$ as follows.

$$\begin{aligned} \mathcal{V}_m &= \left\{ f \in \mathcal{A} : \frac{(\mathrm{z}f'(\mathrm{z}))'}{f'(\mathrm{z})} \in \mathcal{P}_m \right\}, \\ \mathcal{R}_m &= \left\{ f \in \mathcal{A} : \left(\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right) \in \mathcal{P}_m \right\}. \end{aligned}$$

2.8 Convolution

The concept of convolution has proved very powerful in dealing with certain problems of the theory of analytic and univalent functions, particularly closure of families of functions under certain transformations (see [73]). In this section we give the definition of convolution and some other interesting properties which we will use in our forthcoming chapters.

Definition 2.8.1. The convolution or Hadamard product of $f(\mathrm{z}) = \sum_{\ell=0}^{\infty} a_\ell \mathrm{z}^{\ell+1}$ and $g(\mathrm{z}) = \sum_{\ell=0}^{\infty} b_\ell \mathrm{z}^{\ell+1}$ is the function

$$(f * g)(\mathrm{z}) = \sum_{\ell=0}^{\infty} a_\ell b_\ell \mathrm{z}^{\ell+1} = (g * f)(\mathrm{z}), \quad \mathrm{z} \in \mathbb{E}.$$

The function f given by (2.3.1) plays the role of identity function under convolution, that is $(f * g)(\mathrm{z}) = g(\mathrm{z})$ for all g belonging to the class $\mathcal{A}$.

Also we observe that

$$\mathrm{z}(f * g)'(\mathrm{z}) = (\mathrm{z}f' * g)(\mathrm{z}) = (f * \mathrm{z}g')(\mathrm{z}).$$

and

$$(f * k)(\mathrm{z}) = \mathrm{z}f'(\mathrm{z}),$$

where k is Koebe function given in (2.1.2).

The concept of convolution actually arose from the integral

$$h(\mathrm{r}^2 e^{i\theta}) = (f * g)(\mathrm{r}^2 e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\mathrm{r}e^{i(\theta-t)}\right) g(\mathrm{r}e^{it})dt, \quad \mathrm{r} < 1.$$

and the integral convolution is defined by

$$H(\mathrm{z}) = \int_0^{\mathrm{z}} \xi^{-1} h(\xi) d\xi, \quad |\xi| < 1.$$

For detail, see [17].

2.9 Carlson-Shaffer Operator

In 1984, using the idea of convolution, Carlson and Shaffer [7] defined a linear operator $\mathbb{k}(\mathfrak{a}, \mathfrak{c})$ as follows.

Definition 2.9.1. Let $f \in \mathcal{A}$. Then the linear operator $\mathbb{k}(\mathfrak{a}, \mathfrak{c}) : \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathbb{k}(\mathfrak{a}, \mathfrak{c}) f(\mathrm{z}) = \varphi(\mathfrak{a}, \mathfrak{c}; \mathrm{z}) * f(\mathrm{z}) \tag{2.9.1}$$

where

$$\varphi(\mathfrak{a}, \mathfrak{c}; \mathrm{z}) = \mathrm{z} + \sum_{\ell=2}^{\infty} \frac{(\mathfrak{a})_{\ell-1}}{(\mathfrak{c})_{\ell-1}} \mathrm{z}^{\ell}, \qquad \mathfrak{c} \neq 0, -1, -2, ...,$$

is the incomplete beta function with $\varphi(\mathfrak{a}, \mathfrak{c}; \mathrm{z}) \in \mathcal{A}$ and $(\varepsilon)_\ell$ is the Pochhammer symbol defined by

$$(\varepsilon)_\ell = \begin{cases} 1 & \ell = 0 \\ \varepsilon(\varepsilon+1)(\varepsilon+2)...(\varepsilon+\ell-1) & \ell \in \mathbb{N} \end{cases}.$$

In the particular case $\mathfrak{a}=2$, $\mathfrak{c}=1$, the operator $\mathbb{k}(\mathfrak{a},\mathfrak{c})$ reduces to

$$\mathbb{k}(2,1)f(\mathrm{z})=\mathrm{z}f'(\mathrm{z}).$$

Moreover, it is known that

$$\mathrm{z}\left(\mathbb{k}(\mathfrak{a},\mathfrak{c})f(\mathrm{z})\right)'=\mathfrak{a}\left(\mathbb{k}(\mathfrak{a}+1,\mathfrak{c}\right)f(\mathrm{z})-(\mathfrak{a}-1)\mathbb{k}(\mathfrak{a},\mathfrak{c})f(\mathrm{z}).$$

In 1996, with the help of incomplete beta function in multivalent functions

$$\varphi_p(\mathfrak{a},\mathfrak{c};\mathrm{z})=\mathrm{z}^p+\sum_{\ell=1}^{\infty}\frac{(\mathfrak{a})_\ell}{(\mathfrak{c})_\ell}\mathrm{z}^{\ell+p},\quad \left(\mathfrak{a}\in\mathbb{R},\ \mathfrak{c}\in\mathbb{R}\diagdown(0,-1,...)\right),\quad \mathrm{z}\in\mathbb{E}),$$

and convolution, Saitoh [111, 112] introduced the operator $\mathbb{k}_p:\mathcal{A}(p)\to\mathcal{A}(p)$ which is defined as

$$\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})=\mathrm{z}^p+\sum_{\ell=1}^{\infty}\phi_\ell(\mathfrak{a})a_{\ell+p}\mathrm{z}^{\ell+p}$$

with $\mathfrak{a}>-p$ and

$$\phi_\ell(\mathfrak{a})=\frac{\Gamma(\mathfrak{a}+\ell)\Gamma(\mathfrak{c})}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{c}+\ell)}. \tag{2.9.2}$$

This operator is an extension of the familiar Carlson-Shaffer operator given in (2.9.1), see [7, 119]. The following identity can easily be derived

$$\mathrm{z}\left(\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})\right)'=\mathfrak{a}\mathbb{k}_p\left(\mathfrak{a}+1,\mathfrak{c}\right)f(\mathrm{z})-(\mathfrak{a}-p)\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z}). \tag{2.9.3}$$

2.10 Preliminary Results

This section contains several fundamental lemmas, which are essential for the proof of the principal results.

Lemma 2.10.1. [109] Let $h\in\mathcal{P}$ for $\mathrm{z}\in\mathbb{E}$. Then, for $s>0$, $\mu\neq-1\,(\text{complex})$,

$$\mathfrak{Re}\left(h(\mathrm{z})+\frac{\mu_1\mathrm{z}h'(\mathrm{z})}{h(\mathrm{z})+\mu_2}\right)>0,$$

for

$$|z| < \frac{|\mu_2+1|}{\sqrt{A+\sqrt{A^2-|\mu^2-1|^2}}}, \quad A = 2\left(\mu_1+1\right)^2+|\mu_2|^2-1. \tag{2.10.1}$$

This bound is best possible.

Lemma 2.10.2. [115] Let $f \in \mathcal{A}$ and $\frac{f(z)f'(z)}{z} \neq 0$. Then$f$ is in the class of Bazilevic (univalent) functions, iff, for $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and $0 < r < 1$, we have

$$\int_{\theta_1}^{\theta_2}\left[\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}+(\rho-1)\frac{zf'(z)}{f(z)}\right\}-\alpha_1\operatorname{Im}\frac{zf'(z)}{f(z)}\right]d\theta \geq -\pi,$$

where $z = re^{i\theta}$, $\rho > 0$ and α_1 real.

Lemma 2.10.3. [57] Let $u = u_1+iu_2$, $v = v_1+iv_2$ and let $\Psi(u,v)$ be a complex$-$valued function satisfying the conditions.

(i) $\Psi(u,v)$ is continuous in a domain $\mathfrak{D} \subset \mathbb{C}^2$.

(ii) $(1,0) \in \mathfrak{D}$ and $\Psi(1,0) > 0$.

(iii) $\mathfrak{Re}\{\Psi(iu_2,v_1)\} \leq 0$ whenever $(iu_2,v_1) \in \mathfrak{D}$ and $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1+c_1z+c_2z^2+...$, is a function analytic in $\mathbb{E}$ such that $\left(h(z), zh'(z)\right) \in \mathfrak{D}$ and $\mathfrak{Re}\left\{\Psi\left(h(z), zh'(z)\right)\right\} > 0$ for $z \in \mathbb{E}$, then $\mathfrak{Re}h(z) > 0$ in $\mathbb{E}$.

Lemma 2.10.4. [104] Let $f \in \mathcal{S}_\beta$. Then for each β, $|\beta| < \frac{\pi}{2}$, the following sharp inequality holds.

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-2(\cos\beta)r+(\cos 2\beta)r^2}{1-r^2}.$$

Lemma 2.10.5. [41] Let the function w analytic in $\mathbb{E}$ be given by

$$w(z) = c_1z+c_2z^2+... \qquad z \in \mathbb{E}.$$

Then for every complex number μ,

$$\left|c_2-\mu c_1^2\right| \leq 1+(|\mu|-1)\left|c_1^2\right|.$$

Lemma 2.10.6. [31, 74] Let $k \geq 0$ and let δ, σ be any complex numbers with $\delta \neq 0$ and

$0 \leq \delta < R\left(\frac{\delta k}{k+1}+\sigma\right)$. If h is analytic and satisfies

$$\left(h(\mathrm{z})+\frac{\mathrm{z}h^{'}(\mathrm{z})}{\delta h(\mathrm{z})+\sigma}\right) \prec q_{k,\lambda}(\mathrm{z}),$$

and $\tilde{q}_{k,\lambda}(\mathrm{z})$ is a solution of

$$q(\mathrm{z})+\frac{\mathrm{z}\tilde{q}^{'}(\mathrm{z})}{\delta\tilde{q}(\mathrm{z})+\sigma} = q_{k,\lambda}(\mathrm{z}),$$

then $\tilde{q}_{k,\lambda}(\mathrm{z})$ is univalent, and

$$h(\mathrm{z}) \prec \tilde{q}_{k,\lambda}(\mathrm{z}) \prec q_{k,\lambda}(\mathrm{z}).$$

The function $\tilde{q}_{k,\lambda}(\mathrm{z})$ is the best dominant of (2.2.1) and is given as:

$$\tilde{q}_{k,\lambda}(\mathrm{z}) = \left[\int_0^1 \left(\exp\int_t^{tz} \frac{q_{k,\lambda}(u)-1}{u}du\right)dt\right]^{-1} - \frac{\sigma}{\delta}.$$

Lemma 2.10.7. [32] Let $k \in [0,\infty)$ be a fixed number and q_k be the function which belongs to the class $\mathcal{P}(q_k)$. If

$$q_k(\mathrm{z}) = 1+P_1(k)\,\mathrm{z}+P_2(k)\,\mathrm{z}^2+..., \qquad \mathrm{z} \in \mathbb{E},$$

then

$$P_1 := P_1(k) = \begin{cases} \frac{2A^2}{1-k^2}, & 0 \leq k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4Ķ^2(t)^2(1+t)\sqrt{t}} & k > 1. \end{cases} \tag{2.10.2}$$

$$P_2 := P_2(k) = D(k)\,P_1(k)\,, \tag{2.10.3}$$

where

$$D(k) = \begin{cases} \frac{A^2+2}{3}, & 0 \leq k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{(4Ķ(t))^2\left(t^2+6t+1\right)-\pi^2}{24Ķ(t)^2(1+t)\sqrt{t}} & k > 1. \end{cases} \tag{2.10.4}$$

$A = \frac{2}{\pi}\arccos k$, and $\c{K}(t)$ is the complete elliptic integral of first kind (see [33, 34]) .

Lemma 2.10.8. [98] If h is an analytic function with $\mathfrak{Re}h(\mathrm{z}) > 0$ and

$$h(\mathrm{z}) = 1+\sum_{\ell=2}^{\infty} d_\ell \mathrm{z}^\ell, \; \mathrm{z} \in \mathbb{E}, \tag{2.10.5}$$

then, for $\ell \geq 1$,

$$|d_\ell| \leq 2.$$

Lemma 2.10.9. [25] If h is of the form (2.10.5) with positive real part, then

$$\begin{aligned} 2d_2 &= d_1^2+x(4-d_1^2). \\ 4d_3 &= d_1^3+2(4-d_1^2)d_1x-d_1(4-d_1^2)x^2+2(4-d_1^2)\left(1-|x|^2\right)\mathrm{z}. \end{aligned}$$

for some x, z with $|x| \leq 1$ and $|\mathrm{z}| \leq 1$.

Lemma 2.10.10. [19] Let q be convex univalent with $q(0) = 1$ in $\mathbb{E}$ and $\mathfrak{Re}\left(\mu_3 q(\mathrm{z}) + \mu_4\right) > 0$, where $\mu_3, \mu_4 \in \mathbb{C}\diagdown\{0\}$. If h is analytic with $h(0) = q(0)$ in $\mathbb{E}$ and

$$h(\mathrm{z}) + \frac{\mathrm{z}h'(\mathrm{z})}{\mu_3 h(\mathrm{z}) + \mu_4} \prec q(\mathrm{z}), \quad \mathrm{z} \in \mathbb{E},$$

then $h \prec q$.

Lemma 2.10.11. [20] Let $\{d_\ell\}_0^\infty$ be a convex null sequence. Then the function

$$q(\mathrm{z}) = \frac{d_0}{2}+\sum_{\ell=1}^{\infty} d_\ell \mathrm{z}^\ell, \qquad (\mathrm{z} \in \mathbb{E}),$$

is analytic and $\mathfrak{Re}q(\mathrm{z}) > 0$.

Lemma 2.10.12. [46] If ϑ and χ are analytic in E, $\vartheta(0) = \chi(0) = 0$, $\chi(\mathrm{z})$ is starlike in $\mathbb{E}$ and

$$\mathfrak{Re}\left(\frac{\vartheta'(\mathrm{z})}{\chi'(\mathrm{z})}\right) > 0, \quad \text{then} \quad \mathfrak{Re}\left(\frac{\vartheta(\mathrm{z})}{\chi(\mathrm{z})}\right) > 0.$$

Lemma 2.10.13. [117] Let h be given by (2.10.5) with $\mathfrak{Re}h(\mathrm{z}) > \frac{1}{2}$ ($\mathrm{z} \in \mathbb{E}$), then for each function F, analytic in $\mathbb{E}$, the function $h * F$ returns its values in the convex hull of $F(\mathbb{E})$.

Lemma 2.10.14. [48] Let $\beta < 1$. If the function h given by (2.10.5), and

$$\mathfrak{Re}(h(\mathrm{z}){+}\mathrm{z}h^{'}(\mathrm{z})) > \beta, \qquad (\mathrm{z} \in \mathbb{E}).$$

Then

$$\mathfrak{Re}h(\mathrm{z}) > (2\beta{-}1){+}2(1{-}\beta)\ln 2, \qquad (\mathrm{z} \in \mathbb{E}).$$

The result is sharp.

Lemma 2.10.15. [99] For $\alpha \leq 1$ and $\beta \leq 1$,

$$P(\alpha) * P(\beta) \subset P(\delta), \qquad \delta = 1{-}2(1{-}\alpha)(1{-}\beta).$$

The result is sharp.

Lemma 2.10.16. [55] Suppose that the function $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ satisfies the condition $\mathrm{Re}\,(ix, y; z) \leq \delta$ for all real $x, y \leq -\frac{1+x^2}{2}$ and all $\mathrm{z} \in \mathbb{E}$. If h is given by (2.10.5) is analytic and $\mathrm{Re}\,(h(\mathrm{z}), zh'(\mathrm{z}); \mathrm{z}) > \delta$ for $\mathrm{z} \in \mathbb{E}$, then $\mathrm{Re}h(\mathrm{z}) > 0$, $\mathrm{z} \in \mathbb{E}$.

Remark 2.10.1. Throughout this thesis we take h to be analytic in $\mathbb{E}$ with $h(0) = 1$, unless otherwise stated.

Chapter 3

Certain Subclasses of Spirallike Functions Associated with Conic Domains

3.1 Introduction

The classes $k-\mathcal{UCV}$ and $k-\mathcal{ST}$, are the generalization of $\mathcal{C}$ and $\mathcal{S}^*$, were studied in [33, 34] as discussed in section 2.6. Mocanu [61] combined $\mathcal{S}^*$ and $\mathcal{C}$, classes of starlike and convex functions respectively, and defined a new class $\mathcal{M}(\alpha)$, the class of $\alpha-$convex functions mentioned also in section 2.3. Based on the class $\mathcal{M}(\alpha)$, Umarani [124] defined and discussed the class $\mathcal{SC}\left(\alpha,\beta\right)$, the class of $\alpha-$spiral convex functions.
The domain Ω_k, as we discussed in section 2.6, is generalized as follows, see [74]

$$\Omega_{k,\lambda} = (1-\lambda)\,\Omega_k + \lambda, \qquad 0 \leq \lambda < 1.$$

The extremal functions for $\Omega_{k,\lambda}$ are denoted by $q_{k,\lambda}(\mathrm{z})$ with $q_{k,\lambda}(0) = 0$ and $q'_{k,\lambda}(\mathrm{z}) > 0$ which are univalent in $\mathbb{E}$, are given by

$$q_{k,\lambda}\left(\mathrm{z}\right) = \begin{cases} \frac{1+(1-2\lambda)\mathrm{z}}{1-\mathrm{z}}, & k=0, \\ 1+\frac{2(1-\lambda)}{\pi^2}\left(\log\frac{1+\sqrt{\mathrm{z}}}{1-\sqrt{\mathrm{z}}}\right)^2, & k=1, \\ 1+\frac{2(1-\lambda)}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{\mathrm{z}}\right\}, & 0<k<1, \\ 1+\frac{1-\lambda}{k^2-1}\sin\left(\frac{\pi}{2K(t)}\int_0^{\frac{\mu(\mathrm{z})}{t}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(xt)^2}}dx\right)+\frac{1-\lambda}{k^2-1}, & k>1, \end{cases} \tag{3.1.1}$$

where $\mu(\mathrm{z}) = \frac{\mathrm{z}-\sqrt{t}}{1-\sqrt{t\mathrm{z}}}$, $t \in (0,1)$, $\mathrm{z} \in \mathbb{E}$ and z is chosen such that $k = \cosh\left(\pi K'(t)/(4K(t))\right)$, $K(t)$ is the Legendre's complete elliptic integral of the first kind and $K'(t)$ is complementary integral of $K(t)$, see [35]. The class of all functions $q(\mathrm{z})$ analytic in $\mathbb{E}$ and subordinate to $q_{k,\lambda}(\mathrm{z})$ is denoted by $\mathcal{P}(q_{k,\lambda})$. The class $\mathcal{P}(q_{k,\lambda})$ is a convex set [75], also $\mathcal{P}(q_{0,0}) = \mathcal{P}$ and $\mathcal{P}(q_{0,\lambda}) = \mathcal{P}(\lambda)$.
In this chapter, we define certain new classes associated with conic domain $\Omega_{k,\lambda}$. Some results of newly defined classes $k-\mathcal{U}\mathrm{R}^*(p,\beta)$, $k-\mathcal{UC}(p,\beta)$ and $k-\mathcal{UM}^*(p,\alpha,\beta,\gamma)$, such as inclusion properties necessary conditions, are investigated. These classes contain various known classes, such as, β-spirallike functions of order λ, starlike functions, of uniformly

convex (starlike) functions of order λ, convex functions and $\alpha-$spiral convex functions, as special cases.

Definition 3.1.1. Let $f \in \mathcal{A}(p)$. Then $f \in k-\mathcal{U}\mathrm{R}^{*}(p,\beta)$, for $p \in \mathbb{N}$, β real and $|\beta| < \frac{\pi}{2}$, iff

$$\mathfrak{Re}\left\{e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}\right\} > k\left|\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}-p\right| + \lambda\cos\beta, \quad (0 \leq \lambda < 1, \ \ \mathrm{z} \in \mathbb{E}). \tag{3.1.2}$$

Special cases.

(**i**) For $\beta = 0$, we have $k-\mathcal{UR}^{*}(p,0) = \mathcal{UST}_{p}(k,\lambda)$, introduced and studied in [3].

(ii) For $\beta = 0$ and $p = 1$, we have $k-\mathcal{UR}^{*}(1,0) = \mathcal{UST}(k,\lambda)$, defined in [1].

(**iii**) For $k = 0$ and $p = 1$, we have $0-\mathcal{UR}^{*}(1,\beta)$, the class of $\beta-$spirallike functions defined in [24].

(iv) For $k = 0$, $p = 1$ and $\beta = 0$, we have $0-\mathcal{UR}^{*}(1,0) = \mathcal{S}^{*}(\lambda)$, defined in [17, 24].

(v) For $k = 1$, $p = 1$,and $\beta = 0$, we have $1-\mathcal{UR}^{*}(1,0) = 1-\mathcal{UST} \subset \mathcal{S}^{*}(\frac{1}{2})$, defined in [17, 24].

Definition 3.1.2 Let $f \in \mathcal{A}(p)$. Then $f \in k-\mathcal{UC}(p,\beta)$, for $p \in \mathbb{N}$, β real and $|\beta| < \frac{\pi}{2}$, iff

$$\mathfrak{Re}\left\{\frac{e^{i\beta}}{p}\left(1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right\} > k\left|\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}-(p-1)\right| + \lambda\cos\beta, \quad (0 \leq \lambda < 1, \ \ \mathrm{z} \in \mathbb{E}). \tag{3.1.3}$$

Special Cases.

(**i**) For $\beta = 0$, we have $k-\mathcal{UC}(p,0) = \mathcal{UCV}_{p}(k,\lambda)$, introduced and studied in [3].

(**ii**) For $\beta = 0$ and $p = 1$, we have $k-\mathcal{UC}(1,0) = \mathcal{UCV}(k,\lambda)$, introduced and studied in [1].

(**iii**) For $k = 0$ and $p = 1$, we have $0-\mathcal{UC}(1,\beta) = \mathcal{C}^{\beta}(\lambda)$, defined in [101].

(**iv**) For $k = 0$, $p = 1$ and $\beta = 0$, we have $0-\mathcal{UC}(1,0) = \mathcal{C}(\lambda)$, introduced and studied in [17, 24].

We may rewrite the conditions (3.1.2) and (3.1.3) in the form

$$\left\{e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}\right\} \prec q_{k,\lambda}(\mathrm{z}) \quad (0 \leq \lambda < 1,\, p \in \mathbb{N},\; \mathrm{z} \in \mathbb{E}),$$

and

$$\left\{\frac{e^{i\beta}}{p}\left(1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right\} \prec q_{k,\lambda}(\mathrm{z}) \quad (0 \leq \lambda < 1,\, p \in \mathbb{N},\; \mathrm{z} \in \mathbb{E}),$$

where $q_{k,\lambda}$ is given by (3.1.1).

Analogous to the well known Alexander equivalence [17], we have

$$f \in k\text{–}\mathcal{UC}(p,\beta) \qquad \text{iff} \qquad \frac{\mathrm{z}f'}{p} \in k\text{–}\mathcal{UR}^*(p,\beta).$$

Definition 3.1.3. For α, β real , $|\beta| < \frac{\pi}{2}$ and $-\frac{1}{2} \leq \gamma < 1$. Let $f \in \mathcal{A}(p)$ with $\frac{f'(\mathrm{z})f(\mathrm{z})}{p\mathrm{z}} \neq 0$ in $\mathbb{E}$, $p \in \mathbb{N}$ and let

$$\pounds\left(\alpha,\beta,\gamma,f(\mathrm{z})\right) = \left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}+\frac{\alpha\cos\beta}{p(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right). \tag{3.1.4}$$

Then

$$f \in k\text{–}\mathcal{UM}^*(p,\alpha,\beta,\gamma) \qquad \text{iff} \qquad \pounds\left(\alpha,\beta,\gamma,f(\mathrm{z})\right) \prec q_{k,\lambda}(\mathrm{z}) \text{ for } \mathrm{z} \in \mathbb{E}.$$

As special cases, we have the following.

(**i**) For $\lambda = 0$, $0\text{–}\mathcal{UM}^*(1,\alpha,\beta,0) = \mathcal{SC}(\alpha,\beta)$, introduced and studied in [124].

(ii) For $k = 0, p = 1$ and $\alpha = 0$, we have $0\text{–}\mathcal{UM}^*(1,0,\beta,\gamma) = \mathcal{S}^\beta$, studied in [118].

(iii) For $k = 0, p = 1$, $\alpha = 1$ and $\gamma = 0$, we have $0\text{–}\mathcal{UM}^*(1,1,\beta,0) = \mathcal{SC}_\beta$, studied in [128].

(**iv**) For $k = 0, p = 1$, $\beta = 0$ and $\gamma = 0$, we have $0\text{–}\mathcal{UM}^*(1,\alpha,0,0)$, introduced and studied in [55].

(**v**) For $k = 0, p = 1$ and $\gamma = 0$, we have $0\text{–}\mathcal{UM}^*(1,\alpha,\beta,0) = \mathcal{S}_\alpha^\beta(\lambda)$, introduced and studied in [116] (**vi**) For $\lambda = 0$, $p = 1$ and $\beta = \gamma = 0$, we have $0{-}\mathcal{UM}^*(1, \alpha, 0, 0)$, the class of α-starlike functions (of order zero) which has been thoroughly investigated in [58, 60, 61].

3.2 Main Results

Theorem 3.2.1. Let the function $q_{k,0}(\mathrm{z})$ be defined by (2.6.1) and $0 \leq \mathrm{k} < \infty$ be a fixed number. If the function f is a member of the function class $\mathcal{P}\left(q_{k,0}\right)$, then for $-\infty < \mu < \infty$,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| = \begin{cases} \frac{1}{|\Psi_p(\beta)|}\left\{\upsilon \mathrm{P}_1\left(\mathrm{k}\right)^2 - \mathrm{P}_2\left(\mathrm{k}\right)\right\}, & \upsilon > \eta_1\left(\mathrm{k}\right), \\ \frac{P_1(\mathrm{k})}{|\Psi_p(\beta)|}, & \eta_2\left(\mathrm{k}\right) \leq \upsilon \leq \eta_1\left(\mathrm{k}\right), \\ \frac{1}{|\Psi_p(\beta)|}\left(\mathrm{P}_2\left(\mathrm{k}\right) - \upsilon \mathrm{P}_1\left(\mathrm{k}\right)^2\right), & \upsilon < \eta_2\left(\mathrm{k}\right). \end{cases} \tag{3.2.1}$$

where

$$\begin{aligned} \eta_1\left(\mathrm{k}\right) &= \frac{1 + D\left(\mathrm{k}\right)}{\mathrm{P}_1\left(\mathrm{k}\right)}, \\ \eta_2\left(\mathrm{k}\right) &= \frac{D\left(\mathrm{k}\right) - 1}{\mathrm{P}_1\left(\mathrm{k}\right)}, \end{aligned}$$

$$\Psi_p\left(\beta\right) = \frac{\left(e^{i\beta} - \alpha\cos\beta\right)}{p} + \frac{\alpha\cos\beta}{p^2\left(1-\gamma\right)}\left(2 + 7p - p^3\right), \tag{3.2.2}$$

$$\upsilon = \left(\frac{2 + 7p - p^3}{3p + p^2 - p^3}\right)\left(\mu - \frac{2 + 3p + p^2}{2 + 7p - p^3}\right). \tag{3.2.3}$$

and $\mathrm{P}_1\left(\mathrm{k}\right)$, $\mathrm{P}_2\left(\mathrm{k}\right)$ and $D\left(\mathrm{k}\right)$ are given by $(2.10.2)$, $(2.10.3)$ and $(2.10.4)$ respectively.

Proof. Let $f \in \mathrm{k}-\mathcal{UM}^*(p,\alpha,\beta,\gamma)$, then there exists a Schwarz function $w(\mathrm{z})$ in such a way that $w(\mathrm{z}) = q_{\mathrm{k}}(w(\mathrm{z}))$. A simple computation gives

$$\begin{aligned}
a_{p+1} &= \frac{\mathrm{P}_1(\mathrm{k})}{\frac{e^{i\beta}-\alpha\cos\beta}{p}+\frac{\alpha\cos\beta}{p(1-\gamma)}\left(2+p-p^2\right)}c_1, \\
a_{p+2} &= \frac{\mathrm{P}_1\left(\mathrm{k}\right)\left\{c_2+D\left(\mathrm{k}\right)c_1^2\right\}}{\frac{2\left(e^{i\beta}-\alpha\cos\beta\right)}{p}+\frac{\alpha\cos\beta}{p^2(1-\gamma)}\left(2+5p-p^3\right)}+\frac{\left(p^2+3p+2\right)\mathrm{P}_1\left(\mathrm{k}\right)^2c_1^2}{\left(2+7p-p^3\right)\left\{\frac{\left(e^{i\beta}-\alpha\cos\beta\right)}{p}+\frac{\alpha\cos\beta}{p(1-\gamma)}\left(2+p-p^2\right)\right\}^2}.
\end{aligned}$$

Therefore

$$a_{p+2}-\mu a_{p+1}^2 = \frac{\mathrm{P}_1\left(\mathrm{k}\right)}{\Psi_p\left(\beta\right)}\left[c_2+\left\{D\left(\mathrm{k}\right)-\mathrm{P}_1\left(\mathrm{k}\right)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)\right\}c_1^2\right]. \tag{3.2.4}$$

Taking modulus on both sides, we have

$$\left|a_{p+2}-\mu a_{p+1}^2\right| = \frac{\mathrm{P}_1\left(\mathrm{k}\right)}{\left|\Psi_p\left(\beta\right)\right|}\left|c_2-c_1^2+\left\{1+D\left(\mathrm{k}\right)-\mathrm{P}_1\left(\mathrm{k}\right)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)\right\}c_1^2\right|.$$

Suppose that $\upsilon > \eta_1\left(\mathrm{k}\right)$. Then using the estimate $|c_2-c_1^2| \leq 1$ from Lemma 2.10.5 and the known estimate $|c_1| \leq 1$ of the Schwarz Lemma, we have

$$\begin{aligned}
\left|a_{p+2}-\mu a_{p+1}^2\right| &\leq \frac{\mathrm{P}_1\left(\mathrm{k}\right)}{\left|\Psi_p\left(\beta\right)\right|}\left[1+\left\{\mathrm{P}_1\left(\mathrm{k}\right)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)-D\left(\mathrm{k}\right)-1\right\}\right], \\
&= \frac{\mathrm{P}_1\left(\mathrm{k}\right)^2}{\left|\Psi_p\left(\beta\right)\right|}\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)-\frac{\mathrm{P}_2\left(\mathrm{k}\right)}{\left|\Psi_p\left(\beta\right)\right|} \\
&= \frac{1}{\left|\Psi_p\left(\beta\right)\right|}\left\{\upsilon\mathrm{P}_1^2\left(\mathrm{k}\right)-\mathrm{P}_2\left(\mathrm{k}\right)\right\}.
\end{aligned}$$

where $\Psi_p\left(\beta\right)$ and υ is given by (3.2.2) and (3.2.3) respectively. This is the first inequality of (3.2.1).

Now if $\upsilon > \eta_2\left(\mathrm{k}\right)$, then from (3.2.4), we have

$$\left|a_{p+2}-\mu a_{p+1}^2\right| = \frac{\mathrm{P}_1\left(\mathrm{k}\right)}{\left|\Psi_p\left(\beta\right)\right|}\left[\left|c_2\right|+\left\{D\left(\mathrm{k}\right)-\mathrm{P}_1\left(\mathrm{k}\right)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)\right\}\left|c_1\right|^2\right].$$

Applying the estimates $|c_2| \leq 1-c_1^2$ of Lemma 2.10.5 and $|c_1| \leq 1$, we have

$$\begin{aligned} \left|a_{p+2}-\mu a_{p+1}^2\right| &\leq \frac{\mathrm{P}_1(\mathrm{k})}{|\Psi_p(\beta)|}\left[1+\left\{D(\mathrm{k})-\mathrm{P}_1(\mathrm{k})\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)-1\right\}|c_1|^2\right] \\ &\leq \frac{1}{|\Psi_p(\beta)|}\left(\mathrm{P}_2(\mathrm{k})-\upsilon \mathrm{P}_1(\mathrm{k})^2\right). \end{aligned}$$

This is the last expression of (3.2.1).

If $\eta_2(\mathrm{k}) \leq \mu \leq \eta_1(\mathrm{k})$, then

$$\left|D(\mathrm{k})-\mathrm{P}_1(\mathrm{k})\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)\right| \leq 1.$$

Therefore (3.2.4) yields

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{\mathrm{P}_1(\mathrm{k})}{|\Psi_p(\beta)|}\left(|c_2|+|c_1|^2\right) \leq \frac{\mathrm{P}_1(\mathrm{k})}{|\Psi_p(\beta)|}\left(1-|c_1|^2+|c_1|^2\right) = \frac{\mathrm{P}_1(\mathrm{k})}{|\Psi_p(\beta)|}.$$

Thus we have the middle inequality of (3.2.1). $\square$

Theorem 3.2.2. Let $\alpha > 0$, $|\beta| < \frac{\pi}{2}$. Then

$$\mathrm{k}\text{–}\mathcal{UM}^*(p,\alpha,\beta,0) \subset \mathrm{k}\text{–}\mathcal{UR}^*(p,\beta).$$

Further

$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} \prec \tilde{q}_{\mathrm{k},\lambda}(\mathrm{z}) \prec q_{\mathrm{k},\lambda}(\mathrm{z}), \quad \mathrm{z} \in \mathbb{E},$$

where $\tilde{q}_{\mathrm{k},\lambda}(\mathrm{z})$ is the best dominant and is given by

$$\tilde{q}_{\mathrm{k},\lambda}(\mathrm{z}) = \left[\int_0^1 \left(\exp\int_t^{t\mathrm{z}} \frac{q_{\mathrm{k},\lambda}(u)-1}{u}du\right)dt\right]^{-1}. \tag{3.2.5}$$

Proof. Let $f \in \mathrm{k}\text{–}\mathcal{UM}^*(p,\alpha,\beta,0)$ and let

$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} = (\cos\beta)\,h(\mathrm{z})+i\sin\beta = H(\mathrm{z}) \tag{3.2.6}$$

Taking logarithmic differentiation of (3.2.6), and using (3.1.4), we have

$$£\left(\alpha,\beta,f(\mathrm{z})\right)=(\cos\beta)h(\mathrm{z})+i\sin\beta+\frac{\alpha\cos^{2}\beta}{p}\frac{\mathrm{z}h'(\mathrm{z})}{(\cos\beta)h(\mathrm{z})+i\sin\beta}. \tag{3.2.7}$$

From (3.2.7), with (3.2.6) we have

$$H(\mathrm{z})+\frac{1}{\delta}\frac{\mathrm{z}H'(\mathrm{z})}{H(\mathrm{z})}=£\left(\alpha,\beta,f(\mathrm{z})\right)\in \mathrm{k}-U\mathcal{M}^{*}(p,\alpha,\beta). \tag{3.2.8}$$

Applying Lemma 2.10.6, with $\delta=\frac{p}{\alpha\cos\beta}$ and $\sigma=0$, we have

$$H(\mathrm{z})\prec\tilde{q}_{\mathrm{k},\lambda}(\mathrm{z})\prec q_{\mathrm{k},\lambda}(\mathrm{z}),\quad \mathrm{z}\in\mathbb{E},$$

where $\tilde{q}_{\mathrm{k},\lambda}(\mathrm{z})$ is the best dominant of (3.2.8) and is given by (3.2.5). This implies that $f\in \mathrm{k}-\mathcal{UR}^{*}(p,\beta)$. $\square$

If we set $p=1$, $\gamma=0$, $\mathrm{k}=0$ and $\lambda=0$, we get the following result proved in [124].

Corollary 3.2.1. If $f\in\mathcal{SC}(\alpha,\beta)$, then f is $\beta-$spirallike.

Corollary 3.2.2. [116] If $f\in\mathcal{S}_{\alpha}^{\lambda}(\beta)$, $\left(\alpha\geq 0, 0\leq\lambda<1, |\beta|<\frac{\pi}{2}\right)$, then f is $\beta-$spirallike of order λ.

Theorem 3.2.3. Let $f\in\mathcal{A}(p)$. Then $f\in \mathrm{k}-\mathcal{UM}^{*}(p,\alpha,\beta,\gamma)$, iff there exists a function $g\in \mathrm{k}-UR^{*}(p,\beta)$ in such a way that

$$g(\mathrm{z})=f(\mathrm{z})\left(\frac{\mathrm{z}}{f(\mathrm{z})}\right)^{\frac{\alpha\cos\beta}{e^{i\beta}}}\left(f'(\mathrm{z})\right)^{\frac{\alpha\cos\beta}{e^{i\beta}(1-\gamma)}}. \tag{3.2.9}$$

Proof. Let $f\in \mathrm{k}-\mathcal{UM}^{*}(p,\alpha,\beta,\gamma)$, then by Herglotz representation, we have

$$\left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}+\frac{\alpha\cos\beta}{p(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)$$

$$=\cos\beta\int_{0}^{2\pi}\left\{1+\left(\frac{1-2\lambda\mathrm{z}e^{-i\theta}}{1-\mathrm{z}e^{-i\theta}}\right)\right\}d\mu(\theta)+i\sin\beta$$

where $\mu(\theta)$ is a non decreasing function with $\int_{0}^{2\pi}d\mu(\theta)=1$

A simple computation yields

$$\log\frac{f(\mathrm{z})}{\mathrm{z}}\left(\frac{\mathrm{z}}{f(\mathrm{z})}\right)^{\frac{\alpha\cos\beta}{e^{i\beta}}}\left(f'(\mathrm{z})\right)^{\frac{\alpha\cos\beta}{e^{i\beta}(1-\gamma)}}=-\frac{\cos\beta}{e^{i\beta}}(1-2\lambda)\int_0^{2\pi}\log\left(1-\mathrm{z}e^{-i\theta}\right)d\mu(\theta)\quad(3.2.10)$$

Now, there exists some $g\in \mathrm{k}-\mathcal{UR}^*(p,\beta)$ such that the right hand side of (3.2.10) is equal to $\log\frac{g(\mathrm{z})}{\mathrm{z}}$. It follows (3.2.9) .

Conversely suppose that (3.2.9) holds true.

Logarithmic differentiation of (3.2.9) yields

$$\left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}+\frac{\alpha\cos\beta}{p(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)=e^{i\beta}\frac{\mathrm{z}g'(\mathrm{z})}{g(\mathrm{z})}\prec q_{\mathrm{k},\lambda}(\mathrm{z})$$

This implies that $f\in \mathrm{k}-\mathcal{UM}^*(p,\alpha,\beta,\gamma)$. □

Corollary 3.2.3. [124] A function $f\in\mathcal{SC}(\alpha,\beta)$, iff there exists a $\beta-$spiral like function g in such a way that

$$g(\mathrm{z})=f(\mathrm{z})\left[\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right]^{\alpha\cos\beta e^{-i\beta}}.$$

Theorem 3.2.4. A necessary and sufficient condition for the function f to be in $\mathrm{k}-\mathcal{UM}^*(p,\alpha,\beta,\gamma)$, $\alpha\neq 0$ is that f has the integral representation

$$f(\mathrm{z})=\left[\varsigma\int_0^{\mathrm{z}}t^{\varsigma-1}\left(\frac{g(t)}{t}\right)^{\frac{(1-\gamma)e^{i\beta}}{\alpha\cos\beta}}dt\right]^{\frac{1}{\varsigma}},\qquad(3.2.11)$$

for some $g\in \mathrm{k}-UR^*(p,\beta)$, where

$$\varsigma=1+\frac{(1-\gamma)(e^{i\beta}-\alpha\cos\beta)}{\alpha\cos\beta}.\qquad(3.2.12)$$

Proof. Suppose $f \in \mathrm{k}\text{–}\mathcal{UM}^*(p,\alpha,\beta,\gamma)$ and $g \in \mathrm{k}\text{–}\mathcal{UR}^*(p,\beta)$.

From (3.2.9), we have

$$(f(\mathrm{z}))^{\frac{(e^{i\beta}-\alpha\cos\beta)(1-\gamma)}{\alpha\cos\beta}}\left(f(\mathrm{z})\right)' = \left(\frac{g(\mathrm{z})}{\mathrm{z}}\right)^{\frac{e^{i\beta}(1-\gamma)}{\alpha\cos\beta}} \mathrm{z}^{\frac{(1-\gamma)(e^{i\beta}-\alpha\cos\beta)}{\alpha\cos\beta}} \tag{3.2.13}$$

Integrating (3.2.13), from 0 to z, we have (3.2.11), where ς is given by (3.2.12).

Conversely, suppose (3.2.11) holds true with $g \in \mathrm{k}\text{–}\mathcal{UR}^*(p,\beta)$. We have to show that $f \in \mathrm{k}\text{–}\mathcal{UM}^*(p,\alpha,\beta,\gamma)$.

From (3.2.11), after some simplification, we have

$$\left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}+\frac{\alpha\cos\beta}{p(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right) = e^{i\beta}\frac{\mathrm{z}g'(\mathrm{z})}{pg(\mathrm{z})} \prec q_{\mathrm{k},\lambda}(\mathrm{z}),$$

or

$$\left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}+\frac{\alpha\cos\beta}{p(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right) \prec q_{\mathrm{k},\lambda}(\mathrm{z}).$$

This implies that $f \in \mathrm{k}\text{–}\mathcal{UM}^*(p,\alpha,\beta,\gamma)$. $\square$

Theorem 3.2.5. For $0 \le \alpha_1 < \alpha_2$,

$$\mathrm{k}\text{–}\mathcal{UM}^*(p,\alpha_2,\beta,\gamma) \subset \mathrm{k}\text{–}\mathcal{UM}^*(p,\alpha_1,\beta,\gamma).$$

Proof. Let $f \in \mathrm{k}\text{–}\mathcal{UM}^*(p,\alpha_2,\beta,\gamma)$.

Now

$$\begin{aligned}
&\frac{1}{1-\gamma}\left[\left(e^{i\beta}-\alpha_1\cos\beta\right)(1-\gamma)\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}+\frac{\alpha_1\cos\beta}{p}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right] \\
=\;& \frac{\alpha_1}{\alpha_2}\left[\left(e^{i\beta}-\alpha_2\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}+\frac{\alpha_2\cos\beta}{p(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right]-\left(\frac{\alpha_1-\alpha_2}{\alpha_2}\right)e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} \\
=\;& \frac{\alpha_1}{\alpha_2}h_1(\mathrm{z})+\left(1-\frac{\alpha_1}{\alpha_2}\right)h_2(\mathrm{z}) = H(\mathrm{z}),
\end{aligned}$$

where

$$h_1(\mathrm{z}) = \left(e^{i\beta}-\alpha_2\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}+\frac{\alpha_2\cos\beta}{p(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\prec q_{\mathrm{k},\lambda}(\mathrm{z})$$

$$h_2(\mathrm{z}) = e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}\prec q_{\mathrm{k},\lambda}(\mathrm{z}), \qquad \text{(by Theorem 3.2.2)}.$$

Since $\mathcal{P}(q_{\mathrm{k},\lambda})$ is a convex set [75], therefore $H(\mathrm{z})\prec q_{\mathrm{k},\lambda}$. This completes the proof. □

In the following result, we show that the class $\mathrm{k}-\mathcal{UM}^*(p,0,\beta,\gamma)$ is closed under the integral operator, which is a generalized form of Bernardi operator [4] and is given as follows.

$$I_\eta(f) = F(\mathrm{z}) = \frac{\eta+p}{\mathrm{z}^\eta}\int_0^{\mathrm{z}} t^{\eta-1}f(t)dt, \quad \eta>-1. \tag{3.2.14}$$

Theorem 3.2.6. The class $\mathrm{k}-\mathcal{UM}^*(p,0,\beta,\gamma)$ is preserved under the integral operator given in (3.2.14). Further

$$e^{i\beta}\frac{\mathrm{z}F'(\mathrm{z})}{pF(\mathrm{z})}\prec \widetilde{q}_{\mathrm{k},\lambda}(\mathrm{z})\prec q_{\mathrm{k},\lambda}(\mathrm{z}), \quad \mathrm{z}\in\mathbb{E},$$

where $\widetilde{q}_{\mathrm{k},\lambda}(\mathrm{z})$ is the best dominant and is given by

$$\widetilde{q}_{\mathrm{k},\lambda}(\mathrm{z}) = \left[\int_0^1\left(\exp\int_t^{t\mathrm{z}}\frac{q_{\mathrm{k},\lambda}(u)-1}{u}du\right)dt\right]^{-1}-\frac{\eta}{\frac{(\cos\beta+i\sin\beta)}{p}}. \tag{3.2.15}$$

Proof. Set

$$e^{i\beta}\frac{\mathrm{z}F'(\mathrm{z})}{pF(\mathrm{z})} = (\cos\beta)h(\mathrm{z})+i\sin\beta = H(\mathrm{z}), \tag{3.2.16}$$

Then, from (3.2.14), we get

$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} = \cos\beta h(\mathrm{z})+i\sin\beta+\frac{\cos\beta\mathrm{z}h'(\mathrm{z})}{(\cos\beta+i\sin\beta)(\cos\beta h(\mathrm{z})+i\sin\beta)+\eta}. \tag{3.2.17}$$

From (3.2.17) , with (3.2.16), we have

$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} = H(\mathrm{z})+\frac{\mathrm{z}H'(\mathrm{z})}{\delta H(\mathrm{z})+\eta}, \quad \mathrm{z} \in \mathbb{E}, \tag{3.2.18}$$

Applying Lemma 2.10.6, with $\delta = \frac{(\cos\beta+i\sin\beta)}{p}$ and $\sigma = \eta$, we have

$$H(\mathrm{z}) \prec \widetilde{q}_{\mathrm{k},\lambda}(\mathrm{z}) \prec q_{\mathrm{k},\lambda}(\mathrm{z}), \quad \mathrm{z} \in \mathbb{E}.$$

where $\widetilde{q}_{\mathrm{k},\lambda}(\mathrm{z})$ is the best dominant of (3.2.18) and is given by (3.2.15). This implies that $F \in \mathrm{k}-\mathcal{UM}^*(p,0,\beta,\gamma), \quad \mathrm{z} \in \mathbb{E}.$ $\square$

Theorem 3.2.7. A function $f \in \mathcal{A}(p)$ given in (2.4.1) satisfies the condition

$$\left|\frac{1}{e^{i\beta}F(\mathrm{z})}-\frac{1}{2\lambda}\right| < \frac{1}{2\lambda}, \tag{3.2.19}$$

iff, $f \in 0-\mathcal{UR}^*(p,\beta)$, where $F(\mathrm{z}) = \frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}$.

Proof. Suppose f satisfies (3.2.19) . Then we can write

$$\left|\frac{2\lambda-e^{i\beta}F(\mathrm{z})}{2\lambda e^{i\beta}F(\mathrm{z})}\right| < \frac{1}{2\lambda} \quad \text{iff} \quad \left|\frac{2\lambda-e^{i\beta}F(\mathrm{z})}{2\lambda e^{i\beta}F(\mathrm{z})}\right|^2 < \left(\frac{1}{2\lambda}\right)^2$$

$$\text{iff} \quad \left(2\lambda-e^{i\beta}F(\mathrm{z})\right)\left(\overline{2\lambda-e^{i\beta}F(\mathrm{z})}\right) < \left(e^{-i\beta}\overline{F(\mathrm{z})}\right)e^{i\beta}F(\mathrm{z})$$

$$\text{iff} \quad 4\lambda^2 - 2\lambda e^{-i\beta}\overline{F(\mathrm{z})} - 2\lambda e^{i\beta}F(\mathrm{z}) + F(\mathrm{z})\overline{F(\mathrm{z})} < F(\mathrm{z})\overline{F(\mathrm{z})}$$

$$\text{iff} \quad 4\lambda^2 - 2\lambda\left(e^{-i\beta}\overline{F(\mathrm{z})} + e^{i\beta}F(\mathrm{z})\right) < 0$$

$$\text{iff} \quad 2\lambda - 2\mathfrak{Re}\left(e^{i\beta}F(\mathrm{z})\right) < 0$$

$$\text{iff} \quad \mathfrak{Re}\left(e^{i\beta}F(\mathrm{z})\right) > \lambda$$

$$\text{iff} \quad \mathfrak{Re}\left(e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})}\right) > \lambda.$$

This completes the proof. $\square$

Theorem 3.2.8. Let $\alpha \geq 1$, and $|\beta| < \frac{\pi}{2}$. Then

$$\text{k}-\mathcal{UM}^*(p, \alpha, \beta, 0) \subset \mathcal{SC}_\beta.$$

where $\mathcal{SC}_\beta$ denote the class of $\beta-$spirallike convex functions given in (2.3.4).

Proof. Let $f \in \text{k}-\mathcal{UM}^*(p, \alpha, \beta, 0)$, then from (3.2.11) we have

$$f(\text{z}) = \left[\varsigma \int_0^{\text{z}} t^{\varsigma-1}\left(\frac{g(t)}{t}\right)^{\frac{e^{i\beta}}{\alpha\cos\beta}} dt\right]^{\frac{1}{\varsigma}},$$

where ς is given by (3.2.12).

Differentiating the above equation, we obtain

$$\text{z}f'(\text{z})\,(f(\text{z}))^{\varsigma-1} = (g(\text{z}))^{\frac{e^{i\beta}}{\alpha\cos\beta}}. \tag{3.2.20}$$

Logarithmic differentiation of (3.2.20) yields

$$\frac{(\text{z}f'(\text{z}))'}{f'(\text{z})} = (1-\varsigma)\frac{\text{z}f'(\text{z})}{f(\text{z})} + \frac{e^{i\beta}}{\alpha\cos\beta}\frac{\text{z}g'(\text{z})}{g(\text{z})},$$

or, equivalently this can be written as

$$\begin{aligned}
\frac{\cos\beta}{p}\left(\frac{\text{z}f'(\text{z})}{f'(\text{z})}\right)' + i\sin\beta\frac{\text{z}f'(\text{z})}{pf'(\text{z})} &= \left(1-\frac{1}{\alpha}\right)e^{i\beta}\frac{\text{z}f'(\text{z})}{pf'(\text{z})} + \frac{1}{\alpha}e^{i\beta}\frac{\text{z}g'(\text{z})}{pg(\text{z})} \\
&= \left(1-\frac{1}{\alpha}\right)h_3(\text{z}) + \frac{1}{\alpha}h_4(\text{z}) = h(\text{z}),
\end{aligned}$$

where

$$h_3(\text{z}) = e^{i\beta}\frac{\text{z}f'(\text{z})}{pf(\text{z})} \prec q_{\text{k},\lambda}(\text{z}), \quad \text{(by Theorem 3.2.2)},$$

$$h_4(\text{z}) = e^{i\beta}\frac{\text{z}g'(\text{z})}{pg(\text{z})} \prec q_{\text{k},\lambda}(\text{z}), \qquad \text{(by hypothesis)}.$$

Since $\mathcal{P}(q_{\mathrm{k},\lambda})$ is a convex set [75], therefore $h(\mathrm{z}) \prec q_{\mathrm{k},\lambda}(\mathrm{z})$. This completes the proof. $\square$

Theorem 3.2.9. If $f \in \mathcal{A}(p)$ satisfies

$$\sum_{\ell=1}^{\infty} \left\{ \frac{\ell}{p} + \left| \frac{\ell}{p} e^{i\beta} + 2(1-\lambda)\cos\beta \right| \right\} |a_{\ell+p}| < 2(\lambda-1)\cos\beta. \tag{3.2.21}$$

Then $f \in \mathrm{k}-\mathcal{UR}^*(p,\beta)$.

Proof. We assume that the inequality (3.2.21) holds true. It suffices to show that

$$\left| \frac{e^{i\beta}\left(\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} - 1\right)}{e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} - \{(2\lambda-1)\cos\beta + i\sin\beta\}} \right| < 1, \qquad \mathrm{z} \in \mathbb{E}.$$

We have

$$\begin{aligned}
&\left| \frac{e^{i\beta}\left(\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} - 1\right)}{e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} - \{(2\lambda-1)\cos\beta + i\sin\beta\}} \right| \\
= &\left| \frac{e^{i\beta}\sum_{\ell=1}^{\infty}\frac{\ell}{p}a_{\ell+p}\mathrm{z}^{\ell+p}}{\{2(1-\lambda)\cos\beta\}\mathrm{z}^p + \sum_{\ell=1}^{\infty}\left(\frac{\ell}{p}e^{i\beta} + 2(1-\lambda)\cos\beta\right)a_{\ell+p}\mathrm{z}^{\ell+p}} \right| \\
= &\frac{\sum_{\ell=1}^{\infty}\frac{\ell}{p}|a_{\ell+p}|}{|2(\lambda-1)\cos\beta| - \sum_{\ell=1}^{\infty}\left|\frac{\ell}{p}e^{i\beta} + 2(1-\lambda)\cos\beta\right||a_{\ell+p}|}.
\end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{\ell=1}^{\infty}\frac{\ell}{p}|a_{\ell+p}| < 2(\lambda-1)\cos\beta - \sum_{\ell=1}^{\infty}\left|\frac{\ell}{p}e^{i\beta} + 2(1-\lambda)\cos\beta\right||a_{\ell+p}|,$$

which is equivalent to inequality (3.2.21). Hence, we have $f \in \mathrm{k}-\mathcal{UR}^*(p,\beta)$. $\square$

If we set $p = 1$, $\mathrm{k} = 0$ and $\beta = 0$, we have the following result.

Corollary 3.2.4. If $f \in \mathcal{A}$ satisfies

$$\sum_{\ell=1}^{\infty}\{\ell+1-\lambda\}|a_{\ell+1}| < \lambda-1 \qquad \mathrm{z} \in \mathbb{E},$$

then $f \in \mathcal{S}^*(\lambda)$.

Theorem 3.2.10. If $f \in \mathcal{A}(p)$ satisfies

$$\sum_{\ell=1}^{\infty} \left\{\ell+p-1+\left|(\ell+p+1)\, e^{i\beta}-2\lambda\cos\beta\right|\right\} |a_{\ell+p}| < 2p \left\{(\lambda-1)\cos\beta-(p-1)\right\},$$

then $f \in \text{k}-\mathcal{UC}\,(p,\beta)$.

Corollary 3.2.5. If a function $f\,(\text{z}) \in \mathcal{A}$ satisfies

$$\sum_{\ell=1}^{\infty} (\ell+1-2\lambda) < \lambda-1, \qquad \text{z} \in \mathbb{E},$$

then $f \in \mathcal{C}\,(\lambda)$.

3.4 Conclusion

We have generalized the class $\mathcal{SC}\,(\alpha,\beta)$ using k$-$uniformly convexity(starlikeness) and introduced the new subclass of multivalent spirallike functions k$-\mathcal{UM}^*(p,\alpha,\beta,\gamma)$.
We studied some inclusion relations, necessary and sufficient conditions, Fekete-Szego inequality and integral preserving property for these functions. Connection between newly defined class and a number of already known classes of univalent and multivalent functions are highlighted. Assigning particular values to different parameters in our main results, several known results are deduced.

Chapter 4

On λ-Spirallike Analytic Functions Associated with Conic Type Regions

4.1 Introduction

In 1959, Sakaguchi [113] introduced the class $\mathcal{S}_s^*$ of starlike functions with respect to symmetrical points. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_s^*$, iff

$$\frac{2\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})-f(-\mathrm{z})} \prec \frac{1+\mathrm{z}}{1-\mathrm{z}}, \quad (\mathrm{z} \in \mathbb{E}).$$

The class of functions univalent and starlike with respect to symmetrical points includes the classes of convex functions. Also it is shown in [129] that if $f \in \mathcal{S}_s^*$ and $g(\mathrm{z}) = \frac{1}{2}\left[f(\mathrm{z})-f(-\mathrm{z})\right]$, then $g \in \mathcal{S}^*$, the class of starlike functions. From this it follows that $\mathcal{S}_s^* \subset \mathcal{K}$, that is the class of starlike functions with respect to symmetrical points consist of close-to-convex functions.

Motivated by Sakaguchi's work, Das and Singh [14] introduced the class $\mathcal{C}_s$ using the Alexander type relation as:

$$f \in \mathcal{C}_s \quad \text{iff} \quad \mathrm{z}f' \in \mathcal{S}_s^*.$$

Goodman [22, 23] introduced parabolic region as image domain and defined the function which gives exactly parabolic region as its image domain. Kanas and Wisniowska [33, 34] gave hyperbolic and elliptic regions along with their extremal functions, see section 2.6. Then Janowski introduced circular regions and their extremal functions as discussed in section 2.6.2. Recently Noor et.al. [79] combined these(conic and circular) domains and defined a new generalized conic domain $\Omega_{\mathrm{k}}[\mathbb{A}, \mathbb{B}]$ which represents the oval and petal type regions.

In this chapter we define new classes by using the idea of functions belonging to $\mathcal{C}_s$ and $\mathcal{S}_s^*$ associated with generalized conic domain $\Omega_{\mathrm{k}}[\mathbb{A}, \mathbb{B}]$. These generalized classes contains many known classes. The first section serves to introduce some new classes. In the second section, we have some important lemmas which will be used for our further investigation. The remaining section deals our main results.

Definition 4.1.1 [79] A function $h \in$ k-$\mathcal{P}[\mathbb{A},\mathbb{B}]$, iff

$$h(\mathrm{z}) \prec \frac{(\mathbb{A}+1)\, q_{\mathrm{k}}(\mathrm{z}) - (\mathbb{A}-1)}{(\mathbb{B}+1)\, q_{\mathrm{k}}(\mathrm{z}) - (\mathbb{B}-1)}, \quad \mathrm{k} \geq 0, \tag{4.1.1}$$

where $q_{\mathrm{k}}(\mathrm{z})$ is defined in (2.6.1) and $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$.

It can easily be seen that $0-\mathcal{P}[\mathbb{A},\mathbb{B}] \equiv \mathcal{P}[\mathbb{A},\mathbb{B}]$, the class of Janowski functions introduced in [30] and $\mathrm{k}-\mathcal{P}[1,-1] = \mathcal{P}(q_{\mathrm{k}})$, see, [33].

Now using the classes k-$\mathcal{P}[\mathbb{A},\mathbb{B}]$, $\mathcal{S}_s^*$and $\mathcal{C}_s$, we define the following.

Definition 4.1.2. Let $f \in$S. Then $f \in \mathrm{k}-\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$, β is real, $|\beta| < \frac{\pi}{2}$, $\mathrm{k} \geq 0$, $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$, iff

$$e^{i\beta}\frac{2\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z}) - f(-\mathrm{z})} \prec \cos\beta \frac{(\mathbb{A}+1)\, q_{\mathrm{k}}(\mathrm{z}) - (\mathbb{A}-1)}{(\mathbb{B}+1)\, q_{\mathrm{k}}(\mathrm{z}) - (\mathbb{B}-1)} + i\sin\beta, \quad \mathrm{z} \in \mathbb{E}. \tag{4.1.2}$$

Special Cases.

i) For $\mathrm{k} = \beta = 0$, $\mathbb{A} = 1$ and $\mathbb{B} = -1$, we have $0-\mathcal{US}_s^0[1,-1] = \mathcal{S}_s^*$, introduced and studied in [113].

ii) For $\mathrm{k} = \beta = 0$, we have $0-\mathcal{US}_s^0[\mathbb{A},\mathbb{B}] = \mathcal{S}_s^*[\mathbb{A},\mathbb{B}]$, introduced by Goel and Mehrok in 1982 [21].

iii) For $\beta = 0$, $\mathbb{A} = 1$ and $\mathbb{B} = -1$, we have $\mathrm{k}-\mathcal{US}_s^0[1,-1] = \mathrm{k}-\mathcal{ST}_s$, introduced and studied in [66].

Definition 4.1.3. Let $f \in \mathcal{S}$. Then $f \in \mathrm{k}-\mathcal{UC}_s^{\beta}[\mathbb{A},\mathbb{B}]$, β is real, $|\beta| < \frac{\pi}{2}$, $\mathrm{k} \geq 0, -1 \leq \mathbb{B} < \mathbb{A} \leq 1$, iff

$$e^{i\beta}\frac{2(\mathrm{z}f'(\mathrm{z}))'}{(f(\mathrm{z}) - f(-\mathrm{z}))'} \prec \cos\beta \frac{(\mathbb{A}+1)\, q_{\mathrm{k}}(\mathrm{z}) - (\mathbb{A}-1)}{(\mathbb{B}+1)\, q_{\mathrm{k}}(\mathrm{z}) - (\mathbb{B}-1)} + i\sin\beta, \quad \mathrm{z} \in \mathbb{E}.$$

Special Cases.

i) For $\mathrm{k}=\beta=0$, $\mathbb{A}=1$ and $\mathbb{B}=-1$, we have $0-\mathcal{UC}_{s}^{0}[1,-1]$, introduced and studied in [14].

ii) For $\mathrm{k}=\beta=0$, we have $0-\mathcal{UC}_{s}^{0}[\mathbb{A},\mathbb{B}]=\mathcal{C}_{s}^{*}[\mathbb{A},\mathbb{B}]$, introduced by Janteng and Halim in 2008 [29].

iii) For $\beta=0$, $\mathbb{A}=1$ and $\mathbb{B}=-1$, we have $\mathrm{k}-\mathcal{UC}_{s}^{0}[1,-1]=\mathrm{k}-\mathrm{UCV}_{s}$, introduced and studied in [66].

4.2 Preliminary Results

Lemma 4.2.1. Let $\mathrm{k}\in[0,\infty)$ be a fixed and $h_{\mathrm{k}}(\mathrm{z})=\frac{(\mathbb{A}+1)q_{\mathrm{k}}(\mathrm{z})-(\mathbb{A}-1)}{(\mathbb{B}+1)q_{\mathrm{k}}(\mathrm{z})-(\mathbb{B}-1)}$. Then

$$h_{\mathrm{k}}(\mathrm{z})=1+C_{1}(\mathrm{k})\,\mathrm{z}+C_{2}(\mathrm{k})\,\mathrm{z}^{2}+\ldots,\qquad \mathrm{z}\in\mathbb{E}, \tag{4.2.1}$$

and

$$\begin{aligned} C_{1} &:= C_{1}(\mathrm{k})=\frac{\mathbb{A}-\mathbb{B}}{2}P_{1}(\mathrm{k}),\\ C_{2} &:= C_{2}(\mathrm{k})=\frac{1}{4}(\mathbb{A}-\mathbb{B})\left(2D(\mathrm{k})-(\mathbb{B}+1)C_{1}\right)P_{1}(\mathrm{k}). \end{aligned}$$

where $P_{1}(\mathrm{k})$ and $D(\mathrm{k})$ are defined in (2.10.2) and (2.10.4).

Proof. We have

$$\frac{(\mathbb{A}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{A}-1)}{(\mathbb{B}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{B}-1)}=\left[(\mathbb{A}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{A}-1)\right]\left[(\mathbb{B}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{B}-1)\right]^{-1}.$$

Proceeding in a similar way as Noor et.al. [79], we have

$$\begin{aligned}\frac{(\mathbb{A}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{A}-1)}{(\mathbb{B}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{B}-1)} &= \frac{\mathbb{A}-1}{\mathbb{B}-1}+\left(\frac{(\mathbb{A}-1)(\mathbb{B}+1)}{(\mathbb{B}-1)^2}-\frac{\mathbb{A}+1}{\mathbb{B}-1}\right)q_{\mathrm{k}}(\mathrm{z})\\ &\quad+\left(\frac{(\mathbb{A}-1)(\mathbb{B}+1)^2}{(\mathbb{B}-1)^3}-\frac{(\mathbb{A}+1)(\mathbb{B}+1)}{(\mathbb{B}-1)^2}\right)(q_{\mathrm{k}}(\mathrm{z}))^2\\ &\quad+\left(\frac{(\mathbb{A}-1)(\mathbb{B}+1)^3}{(\mathbb{B}-1)^4}-\frac{(\mathbb{A}+1)(\mathbb{B}+1)^2}{(\mathbb{B}-1)^3}\right)(q_{\mathrm{k}}(\mathrm{z}))^3+...\end{aligned}$$

If $q_{\mathrm{k}}(\mathrm{z})=1+P_1(\mathrm{k})\,\mathrm{z}+P_2(\mathrm{k})\,\mathrm{z}^2+\ldots$, then after suitable simplifications

$$\begin{aligned}\frac{(\mathbb{A}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{A}-1)}{(\mathbb{B}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{B}-1)} &= \sum_{\ell=2}^{\infty}\frac{-2(\mathbb{B}+1)^{\ell-1}}{(\mathbb{B}-1)^{\ell}}+\left\{\sum_{\ell=2}^{\infty}\frac{2\ell(\mathbb{A}-\mathbb{B})(\mathbb{B}+1)^{\ell-1}}{(\mathbb{B}-1)^{\ell+1}}\right\}P_1(\mathrm{k})\,\mathrm{z}\\ &\quad+\left\{\sum_{\ell=2}^{\infty}\frac{2(\mathbb{A}-\mathbb{B})(\mathbb{B}+1)^{\ell-1}}{\frac{1}{2}\ell D(\mathrm{k})(\ell-1)\,C_1(\mathrm{k})(\mathbb{B}-1)^{\ell+1}}\right\}P_1\mathrm{z}^2+...\end{aligned}$$

Now the series

$$\sum_{\ell=2}^{\infty}\frac{-2(\mathbb{B}+1)^{\ell-1}}{(\mathbb{B}-1)^{\ell}},\ \sum_{\ell=2}^{\infty}\frac{2\ell(\mathbb{A}-\mathbb{B})(\mathbb{B}+1)^{\ell-1}}{(\mathbb{B}-1)^{\ell+1}}\ \text{and}\ \sum_{\ell=2}^{\infty}\frac{2(\mathbb{A}-\mathbb{B})(\mathbb{B}+1)^{\ell-1}}{\frac{1}{2}\ell D(\mathrm{k})(\ell-1)\,C_1(\mathrm{k})(\mathbb{B}-1)^{\ell+1}}$$

are convergent and converge to 1, $\frac{\mathbb{A}-\mathbb{B}}{2}$ and $\frac{(\mathbb{A}-\mathbb{B})(2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k}))}{4}$ respectively. Thus we have

$$\begin{aligned}\frac{(\mathbb{A}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{A}-1)}{(\mathbb{B}+1)\,q_{\mathrm{k}}(\mathrm{z})-(\mathbb{B}-1)} &= 1+\frac{1}{2}(\mathbb{A}-\mathbb{B})\,P_1(\mathrm{k})\,\mathrm{z}+\frac{1}{4}\{(\mathbb{A}-\mathbb{B})(2D(\mathrm{k})-(\mathbb{B}+1)\,P_1(\mathrm{k}))\}\\ &\quad\times P_1(\mathrm{k})\,\mathrm{z}^2+....\end{aligned}$$

This completes the proof. $\square$

4.3 Main Results

Theorem 4.3.1. Let $f \in \mathrm{k}-\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$, $|\beta| < \frac{\pi}{2}$, $\mathrm{k} \geq 0$ and $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$. Then the function

$$\varphi(\mathrm{z}) = \frac{1}{2}\left(f(\mathrm{z}) - f(-\mathrm{z})\right), \tag{4.3.1}$$

belongs to $\mathrm{k}-\mathcal{US}[\mathbb{A},\mathbb{B}]$ in E, where $\mathrm{k}-\mathcal{US}[\mathbb{A},\mathbb{B}]$ is the class of Janowski starlike functions related with conic type region $\mathrm{k}-\mathcal{P}[\mathbb{A},\mathbb{B}]$.

Proof. Taking logarithmic differentiation of (4.3.1) it follows that

$$\frac{\mathrm{z}\varphi'(\mathrm{z})}{\varphi(\mathrm{z})} = \frac{\mathrm{z}\left(f(\mathrm{z})\right)' + \mathrm{z}\left(f(-\mathrm{z})\right)'}{\left(f(\mathrm{z}) - f(-\mathrm{z})\right)}.$$

We have

$$\begin{aligned} \frac{\mathrm{z}\varphi'(\mathrm{z})}{\varphi(\mathrm{z})} &= \frac{1}{2}\left[\frac{2\mathrm{z}\left(f(\mathrm{z})\right)'}{\left(f(\mathrm{z}) - f(-\mathrm{z})\right)} + \frac{2\mathrm{z}\left(f(-\mathrm{z})\right)'}{\left(f(-\mathrm{z}) - f(\mathrm{z})\right)}\right] \\ &= \frac{1}{2}\left[h_1(\mathrm{z}) + h_2(\mathrm{z})\right], \quad \text{for } \mathrm{z} \in \mathbb{E},\ h_1, h_2 \in \mathrm{k}-\mathcal{P}[\mathbb{A},\mathbb{B}]. \end{aligned}$$

Since $\mathrm{k}-\mathcal{P}[\mathbb{A},\mathbb{B}]$ is a convex set, it follows that $\frac{\mathrm{z}\varphi'(\mathrm{z})}{\varphi(\mathrm{z})} \in \mathrm{k}-\mathcal{P}[\mathbb{A},\mathbb{B}]$ and thus $\varphi \in \mathrm{k}-\mathcal{US}[\mathbb{A},\mathbb{B}]$. $\square$

The above Theorem shows that the class $\mathrm{k}-\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$ is a subclass of $\mathcal{K}$ of close-to-convex functions.

Theorem 4.3.2. Let $\mathrm{k} \in [0,\infty]$ be fixed. Let $h(\mathrm{z}) = 1 + d_1\mathrm{z} + d_2\mathrm{z}^2 + \ldots$, represented in definition 4.2.1 belong to the class $\mathrm{k}-\mathcal{P}[\mathbb{A},\mathbb{B}]$. Then for $-\infty < u < \infty$,

$$\left|d_2 - ud_1^2\right| \leq \begin{cases} \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\left\{u\frac{(\mathbb{A}-\mathbb{B})}{2}P_1(\mathrm{k}) - \frac{1}{2}\left\{(2D(\mathrm{k}) - (\mathbb{B}+1)P_1(\mathrm{k}))\right\}\right\}, & u > \alpha_1 \\ \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k}), & \alpha_1 \leq u \leq \alpha_2, \\ \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\left\{\frac{1}{2}\left\{(2D(\mathrm{k}) - (\mathbb{B}+1)P_1(\mathrm{k}))\right\} - u\frac{(\mathbb{A}-\mathbb{B})}{2}P_1(\mathrm{k})\right\}, & u < \alpha_2, \end{cases} \tag{4.3.2}$$

where

$$\alpha_1 = \frac{2+2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k})}{(\mathbb{A}-\mathbb{B})T_1(\mathrm{k})},$$
$$\alpha_2 = \frac{2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k})-2}{(\mathbb{A}-\mathbb{B})P_1(\mathrm{k})},$$

and P_1, $D(\mathrm{k})$ are defined in (2.10.2) and (2.10.4).

Proof. If $h \in \mathrm{k}-\mathcal{P}[\mathbb{A},\mathbb{B}]$ then it follows that

$$h(\mathrm{z}) \prec q_{\mathrm{k}}(\mathrm{z}) = 1+\frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\,\mathrm{z}+\frac{(\mathbb{A}-\mathbb{B})(2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k}))}{4}P_1(\mathrm{k})\,\mathrm{z}^2+...., \quad \mathrm{z} \in \mathbb{E}. \tag{4.3.3}$$

Now by the definition of subordination, there exists a function $w(\mathrm{z})$ analytic in E with $w(0) = 0$ and $|w(\mathrm{z})| < 1$ such that

$$h(\mathrm{z}) = 1+\frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\,w(\mathrm{z})+\frac{(\mathbb{A}-\mathbb{B})(2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k}))}{4}P_1(\mathrm{k})\,w^2(\mathrm{z})+.... \tag{4.3.4}$$

From (4.3.3), (4.3.4) and using Lemma 2.10.5, we have

$$d_1 = \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\,c_1,$$
$$d_2 = \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\left\{c_2+\frac{(2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k}))}{2}c_1^2\right\}.$$

Therefore

$$d_2-ud_1^2 = \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\left\{c_2+\left\{\frac{(2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k}))}{2}-u\frac{(\mathbb{A}-\mathbb{B})}{2}P_1(\mathrm{k})\right\}c_1^2\right\}. \tag{4.3.5}$$

This gives us

$$\left|d_2-ud_1^2\right| = \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\left|c_2-c_1^2+\left\{1+\frac{(2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k}))}{2}-u\frac{(\mathbb{A}-\mathbb{B})}{2}P_1(\mathrm{k})\right\}c_1^2\right|.$$

Suppose that $u > \alpha_1$, then using the estimate $|c_2-c_1^2| \leq 1$ from Lemma 2.10.5 and the well known estimate $|c_1| \leq 1$ of the Schwarz Lemma, we obtain

$$\left|d_2 - ud_1^2\right| \leq \frac{\mathbb{A}-\mathbb{B}}{2} P_1(\mathrm{k}) \left\{ u\frac{(\mathbb{A}-\mathbb{B})}{2} P_1(\mathrm{k}) - \frac{(2D(\mathrm{k}) - (\mathbb{B}+1) P_1(\mathrm{k}))}{2} \right\}.$$

This is the first inequality in (4.3.2).

On the other hand if $u < \alpha_2$, then (4.3.5) gives

$$\left|d_2 - ud_1^2\right| \leq \frac{\mathbb{A}-\mathbb{B}}{2} P_1(\mathrm{k}) \left\{ |c_2| + \left\{ \frac{(2D(\mathrm{k}) - (\mathbb{B}+1) P_1(\mathrm{k}))}{2} - u\frac{(\mathbb{A}-\mathbb{B})}{2} P_1(\mathrm{k}) \right\} |c_1|^2 \right\}.$$

Applying the estimates $|c_2| \leq 1 - |c_1|^2$ of Lemma 2.10.5 and $|c_1| \leq 1$, we have

$$\begin{aligned} \left|d_2 - ud_1^2\right| &\leq \frac{\mathbb{A}-\mathbb{B}}{2} P_1(\mathrm{k}) \left\{ 1 + \left\{ \frac{(2D(\mathrm{k}) - (\mathbb{B}+1) P_1(\mathrm{k}))}{2} - u\frac{(\mathbb{A}-\mathbb{B})}{2} P_1(\mathrm{k}) - 1 \right\} |c_1|^2 \right\} \\ &\leq \frac{\mathbb{A}-\mathbb{B}}{2} P_1(\mathrm{k}) \left\{ \frac{(2D(\mathrm{k}) - (\mathbb{B}+1) P_1(\mathrm{k}))}{2} - u\frac{(\mathbb{A}-\mathbb{B})}{2} P_1(\mathrm{k}) \right\}. \end{aligned}$$

This is the last inequality in (4.3.2). Lastly if $\alpha_1 \leq u \leq \alpha_2$, then

$$\left| \frac{(2D(\mathrm{k}) - (\mathbb{B}+1) P_1(\mathrm{k}))}{2} - u\frac{(\mathbb{A}-\mathbb{B})}{2} P_1(\mathrm{k}) \right| \leq 1.$$

Therefore (4.3.5) , yields

$$\left|d_2 - ud_1^2\right| \leq \frac{\mathbb{A}-\mathbb{B}}{2} P_1(\mathrm{k}) \left\{ |c_2| + |c_1|^2 \right\} \leq \frac{\mathbb{A}-\mathbb{B}}{2} P_1(\mathrm{k}) \left\{ 1 - |c_1|^2 + |c_1|^2 \right\} = \frac{\mathbb{A}-\mathbb{B}}{2} P_1(\mathrm{k}),$$

and we get the middle inequality in (4.3.2). This completes the proof. □

Remark 4.3.1. In above Theorem if we set $\mathbb{A} = 1$ and $\mathbb{B} = -1$ we have the result given in [59].

Theorem 4.3.3. Let $f \in \mathcal{A}$. Then $f \in \mathrm{k}-\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$, $0 \leq \mathrm{k} < \infty$, $|\beta| < \frac{\pi}{2}$, $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$. Then

$$\left|\mu a_2^2 - a_3\right| \leq \frac{1}{2}\begin{cases} \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\left\{\frac{\mu\cos\beta e^{-i\beta}(\mathbb{A}-\mathbb{B})}{4}P_1(\mathrm{k}) - \frac{(2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k}))}{2}\right\}, & \mu > \delta_1, \\ \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k}), & \delta_1 \leq \mu \leq \delta_2, \\ \frac{\mathbb{A}-\mathbb{B}}{2}P_1(\mathrm{k})\left\{\frac{(2D(\mathrm{k})-(\mathbb{B}+1)P_1(\mathrm{k}))}{2} - \frac{\mu\cos\beta e^{-i\beta}(\mathbb{A}-\mathbb{B})}{4}P_1(\mathrm{k})\right\}, & \mu < \delta_2, \end{cases}$$

where

$$\begin{aligned} \delta_1 &= \frac{2\left(2+2D(\mathrm{k}) - (\mathbb{B}+1)P_1(\mathrm{k})\right)}{(\mathbb{A}-\mathbb{B})P_1(\mathrm{k})\cos\beta}e^{i\beta}, \\ \delta_1 &= \frac{2\left(2D(\mathrm{k}) - (\mathbb{B}+1)P_1(\mathrm{k}) - 2\right)}{(\mathbb{A}-\mathbb{B})P_1(\mathrm{k})\cos\beta}e^{i\beta}. \end{aligned}$$

and P_1, $D(\mathrm{k})$ are defined in $(2.10.2)$ and $(2.10.4)$.

Proof. By definition of the class $\mathrm{k}-\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$, there exists a function $h \in \mathcal{S}$, represented by $h(\mathrm{z}) = 1+d_1\mathrm{z}+d_2\mathrm{z}^2+...$ and subordinate to q_{k}, given by $(4.2.1)$, such that

$$e^{i\beta}\frac{2\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z}) - f(-\mathrm{z})} = \cos\beta h(\mathrm{z}) + i\sin\beta, \quad \mathrm{z} \in \mathbb{E}.$$

Substituting the corresponding series expansions and by equating coefficients we obtain

$$\begin{aligned} a_2 &= \frac{1}{2}d_1\cos\beta e^{-i\beta}, \\ a_3 &= \frac{1}{2}d_2\cos\beta e^{-i\beta}. \end{aligned}$$

Therefore

$$\left|\mu a_2^2 - a_3\right| \leq \frac{1}{2}\left|\frac{\mu d_1^2\cos\beta e^{-i\beta}}{2} - d_2\right|.$$

An application of Theorem 4.3.2, gives the required result. This completes the proof. $\square$

Theorem 4.3.4. For $\mathrm{k} \geq 0$, $|\beta| < \frac{\pi}{2}$, $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$. A function $f \in \mathrm{k}-\mathcal{US}_s^{\beta}[\mathbb{A}, \mathbb{B}]$, iff

$$\frac{1}{\mathrm{z}}\left\{f(\mathrm{z}) * \left[\frac{\mathrm{z}-M_\beta \mathrm{z}^2}{(1-\mathrm{z})^2(1+\mathrm{z})}\right]\right\} \neq 0, \qquad \mathrm{z} \in \mathbb{E}, \quad 0 \leq \theta < 2\pi, \tag{4.3.6}$$

for all values of M_β, where

$$M_\beta = \frac{\left(1+e^{-i\beta} i\sin\beta\right)\left((\mathbb{B}+1)\, q_{\mathrm{k}}\left(e^{i\theta}\right) - (\mathbb{B}-1)\right) + e^{-i\beta}\cos\beta\left\{(\mathbb{A}+1)\, q_{\mathrm{k}}\left(e^{i\theta}\right) - (\mathbb{A}-1)\right\}}{e^{-i\beta}\cos\beta\left\{(\mathbb{A}+1)\, q_{\mathrm{k}}\left(e^{i\theta}\right) + (\mathbb{A}-1)\right\} - \left(1-e^{-i\beta} i\sin\beta\right)\left((\mathbb{B}+1)\, q_{\mathrm{k}}\left(e^{i\theta}\right) + (\mathbb{B}-1)\right)}. \tag{4.3.7}$$

Proof. Since $f \in \mathrm{k}-\mathcal{US}_s^{\beta}[\mathbb{A}, \mathbb{B}]$ is analytic in E, it follows that $(f(\mathrm{z})-f(-\mathrm{z}))/2 \neq 0$, for all z $\in$E. Using the fact

$$f(\mathrm{z}) * \frac{\mathrm{z}}{(1-\mathrm{z}^2)} = \frac{1}{2}\left[f(\mathrm{z})-f(-\mathrm{z})\right], \qquad \mathrm{z} \in \mathbb{E},$$

we obtain the desired result for $M_\beta = 1$.

Since f is an arbitrary function, then from (4.1.2), we have

$$e^{i\beta}\frac{2\mathrm{z}f'(\mathrm{z})}{[f(\mathrm{z})-f(-\mathrm{z})]} \prec \cos\beta\frac{(\mathbb{A}+1)\, q_{\mathrm{k}}(\mathrm{z}) - (\mathbb{A}-1)}{(\mathbb{B}+1)\, q_{\mathrm{k}}(\mathrm{z}) - (\mathbb{B}-1)} + i\sin\beta, \quad \mathrm{z} \in \mathbb{E}. \tag{4.3.8}$$

From (4.3.8), according to the definition of the subordination, there exists a Schwarz function w such that

$$e^{i\beta}\frac{2\mathrm{z}f'(\mathrm{z})}{[f(\mathrm{z})-f(-\mathrm{z})]} = \cos\beta\frac{(\mathbb{A}+1)\, q_{\mathrm{k}}(w(\mathrm{z})) - (\mathbb{A}-1)}{(\mathbb{B}+1)\, q_{\mathrm{k}}(w(\mathrm{z})) - (\mathbb{B}-1)} + i\sin\beta, \qquad \mathrm{z} \in \mathbb{E}.$$

A simple computation gives

$$\frac{1}{\mathrm{z}}\left\{\begin{array}{c} \mathrm{z}f'(\mathrm{z})\left[(\mathbb{B}+1)\, q_{\mathrm{k}}(e^{i\theta}) - (\mathbb{B}-1)\right] - \frac{1}{2}\left[f(\mathrm{z})-f(-\mathrm{z})\right] e^{-i\beta}\times \\ \left[\cos\beta\left((\mathbb{A}+1)\, q_{\mathrm{k}}(e^{i\theta}) - (\mathbb{A}-1)\right) + i\sin\beta\left((\mathbb{B}+1)\, q_{\mathrm{k}}(e^{i\theta}) - (\mathbb{B}-1)\right)\right] \end{array}\right\} \neq 0, \tag{4.3.9}$$

for $z \in \mathbb{E}$, $\theta \in [0, 2\pi)$. Using the convolution properties

$$f(z) * \frac{z}{(1-z)^2} = zf'(z) \quad \text{and} \quad f(z) * \frac{z}{(1-z^2)} = \frac{1}{2}[f(z)-f(-z)], \qquad z \in \mathbb{E}$$

we have that

$$\frac{1}{z}\left\{\begin{array}{l} f(z) * \left[\frac{z\left[(\mathbb{B}+1)q_k(e^{i\theta})-(\mathbb{B}-1)\right]}{(1-z)^2}\right. \\ \left. -\frac{ze^{-i\beta}\left[\cos\beta\left((\mathbb{A}+1)q_k(e^{i\theta})-(\mathbb{A}-1)\right)+i\sin\beta\left((\mathbb{B}+1)q_k(e^{i\theta})-(\mathbb{B}-1)\right)\right]}{1-z^2}\right] \end{array}\right\} \neq 0$$

Hence it follows that

$$\frac{1}{z}\left\{f(z) * \left[\frac{\left[z+\left(\frac{\left(1+e^{-i\beta}i\sin\beta\right)\left((\mathbb{B}+1)q_k\left(e^{i\theta}\right)-(\mathbb{B}-1)\right)+e^{-i\beta}\cos\beta\left\{(\mathbb{A}+1)q_k\left(e^{i\theta}\right)-(\mathbb{A}-1)\right\}}{\left(1-e^{-i\beta}i\sin\beta\right)\left((\mathbb{B}+1)q_k\left(e^{i\theta}\right)-(\mathbb{B}-1)\right)-e^{-i\beta}\cos\beta\left\{(\mathbb{A}+1)q_k\left(e^{i\theta}\right)-(\mathbb{A}-1)\right\}}\right)z^2\right]}{(1-z)^2(1+z)}\right]\right\} \neq 0 \tag{4.3.10}$$

for $z \in \mathbb{E}$, $\theta \in [0, 2\pi)$, which is the required condition.

Conversely, suppose that the condition (4.3.6) holds for $M_\beta = 1$, it follows that $\frac{z}{2}(f(z)-f(-z)) \neq 0$, $z \in \mathbb{E}$. Thus the function $h(z) = \frac{2zf'(z)}{(f(z)-f(-z))}$ is analytic in $\mathbb{E}$. Since we have shown that (4.3.10) and (4.3.9) are equivalent, therefore, we have

$$e^{i\beta}\frac{2zf'(z)}{f(z)-f(-z)} \neq \cos\beta\frac{(\mathbb{A}+1)q_k(e^{i\theta})-(\mathbb{A}-1)}{(\mathbb{B}+1)q_k(e^{i\theta})-(\mathbb{B}-1)}+i\sin\beta, \quad z \in \mathbb{E}. \tag{4.3.11}$$

Suppose that

$$H(z) = \cos\beta\frac{(\mathbb{A}+1)q_k(z)-(\mathbb{A}-1)}{(\mathbb{B}+1)q_k(z)-(\mathbb{B}-1)}+i\sin\beta, \quad z \in \mathbb{E}.$$

Now, from relation (4.3.11) it is clear that $H(\partial\mathbb{E}) \cap h(\mathbb{E}) = \phi$. Therefore, the simply connected domain $h(\mathbb{E})$ is contained in a connected component of $\mathbb{C}\backslash H(\partial\mathbb{E})$. The univalence of the function h together with the fact $H(0) = h(0) = 1$ shows that $h \prec H$ which shows that $f \in \text{k}-\mathcal{US}_s^\beta[\mathbb{A}, \mathbb{B}]$.

From Theorem 4.3.4, we obtain the following special cases.

Corollary 4.3.1. For $\beta = 0, -1 \leq \mathbb{B} < \mathbb{A} \leq 1$. A function $f \in \mathrm{k}-\mathcal{US}_s^0[\mathbb{A}, \mathbb{B}]$, iff

$$\frac{1}{\mathrm{z}}\left\{f(\mathrm{z}) * \left[\frac{\mathrm{z}+\frac{\left[(\mathbb{B}+\mathbb{A}+2)q_{\mathrm{k}}(e^{i\theta})-(\mathbb{B}+\mathbb{A}-2)\right]}{(\mathbb{B}-\mathbb{A})\left(q_{\mathrm{k}}(e^{i\theta})-1\right)}\mathrm{z}^2}{(1-\mathrm{z})^2(1+\mathrm{z})}\right]\right\} \neq 0, \quad \mathrm{z} \in \mathbb{E}, \quad 0 \leq \theta < 2\pi.$$

Corollary 4.3.2. A function $f \in 0-\mathcal{US}_s^0[1, -1]$, iff

$$\frac{1}{\mathrm{z}}\left\{f(\mathrm{z}) * \left[\frac{\mathrm{z}\left(1-\mathrm{z}e^{-i\theta}\right)}{(1-\mathrm{z})^2(1+\mathrm{z})}\right]\right\} \neq 0, \quad \mathrm{z} \in \mathbb{E}, \quad 0 \leq \theta < 2\pi.$$

Theorem 4.3.5. If $f \in \mathcal{S}$, then $f \in \mathrm{k}-\mathcal{UC}_s^\beta[\mathbb{A}, \mathbb{B}]$, iff

$$\frac{1}{\mathrm{z}}\left\{f(\mathrm{z}) * \frac{1+2\mathrm{z}^3+[M_\beta-3]\mathrm{z}^2-3M_\beta\mathrm{z}^4}{\mathrm{z}(1-\mathrm{z})^3(1+\mathrm{z})^2}\right\} \neq 0, \quad \mathrm{z} \in \mathbb{E}, \quad 0 \leq \theta < 2\pi,$$

where $-1 \leq \mathbb{B} < \mathbb{A} \leq 1, \mathrm{k} \geq 0$, $|\beta| < \frac{\pi}{2}$ and M_β is given in (4.3.7).

Proof. Let

$$g(\mathrm{z}) = \frac{\mathrm{z}+M_\beta\mathrm{z}^2}{(1-\mathrm{z})^2(1+\mathrm{z})}.$$

then

$$\mathrm{z}g'(\mathrm{z}) = \frac{\mathrm{z}+M_\beta\mathrm{z}^4+(M_\beta+2)\mathrm{z}^3+(2M_\beta+1)\mathrm{z}^2}{(1-\mathrm{z})^3(1+\mathrm{z})^2}.$$

Now using the Alexander type relation between $\mathrm{k}-\mathcal{US}_s^\beta[\mathbb{A}, \mathbb{B}]$ and $\mathrm{k}-\mathcal{UC}_s^\beta[\mathbb{A}, \mathbb{B}]$, the identity

$$\mathrm{z}f'(\mathrm{z}) * g(\mathrm{z}) = f(\mathrm{z}) * \mathrm{z}g'(\mathrm{z}),$$

and Theorem 4.3.4, we obtain the required result. $\square$

4.4 Conclusion

In this chapter, we have defined some new subclasses of spiralike analytic functions. The convolution properties and Fekete-Szego inequalities for these classes are studied. The relation between newly defined classes and previously known ones are established. Several known results have also been deduced from our main results as special cases by assigning particular values to different parameters.

Chapter 5

On Generalized Spirallike Analytic Functions

5.1 Introduction

The classes $\mathcal{V}_m$ and $\mathcal{R}_m$ are discussed in section 2.7.1 and the $\beta-$spirallike functions in section 2.3.1. Specek [118] generalized the notion of starlike functions and gave the definition of $\beta-$spirallike functions. After that, Libera [44] extended this definition to functions spirallike of order λ denoted by $\mathcal{S}_\beta(\lambda)$. In 1969 Silvia [116] defined a class $\mathcal{M}(\alpha, \beta, \lambda)$ which contains the classes of Mocanu variation and β-spirallike functions of order λ as special cases discussed in section 2.3.
In this chapter, Using the functions with bounded boundary and bounded radius rotation we will study the classes $\mathcal{R}_m^*(\beta), \mathcal{M}_m^*(\alpha, \beta)$ and $\mathcal{B}_m(\alpha, \beta, \gamma)$ of generalized starlike, spirallike and Mocanu functions [61] respectively. In the first section we give definitions and in the remaining sections, we investigate some properties of these classes.

Definition 5.1.1. Let $f \in \mathcal{A}$ given in (2.1.1). Then, for β real and $|\beta| < \frac{\pi}{2}$, $f \in \mathcal{R}_m^*(\beta)$ iff

$$\left\{e^{i\beta}\frac{\mathrm{z}f(\mathrm{z})'}{f(\mathrm{z})}\right\} \in \mathcal{P}_m \quad \mathrm{z} \in \mathbb{E},\ m \geq 2.$$

We note that $\mathcal{R}_m^*(0) = \mathcal{R}_m$ the class of functions of bounded radius rotation, see [24] and $\mathcal{R}_2^*(\beta) = \mathcal{S}^*(\beta)$ and $\mathcal{R}_2^*(0) = \mathcal{S}^*$.
Definition 5.1.2. Let $f \in \mathcal{A}$ with $\frac{f(\mathrm{z})f'(\mathrm{z})}{\mathrm{z}} \neq 0$ in $\mathbb{E}$, and let

$$\mathcal{J}_m(\alpha, \beta, f(\mathrm{z})) = \left(e^{i\beta} - \alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} + \alpha\cos\beta\left(1 + \frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right).$$

Then

$$f(\mathrm{z}) \in \mathcal{M}_m^*(\alpha, \beta) \text{ iff } \mathcal{J}_m(\alpha, \beta, f(\mathrm{z})) \in \mathcal{P}_m, \quad \mathrm{z} \in \mathbb{E},\ \alpha, \beta \text{ real and } |\beta| < \frac{\pi}{2}.$$

We note, as a special case, that the class $\mathcal{M}_m^*(\alpha, 0)$ coincides with the class of Mocanu variation, see [13].

Definition 5.1.3. Let $f \in \mathcal{A}$ be given by (2.1.1) with $\frac{f(\mathrm{z})f'(\mathrm{z})}{\mathrm{z}} \neq 0$ in $\mathbb{E}$, and let

$$\tilde{\mathcal{J}}\left(\alpha,\beta,\gamma,f(\mathrm{z})\right)=\left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\frac{\alpha\cos\beta}{(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right),$$

for real $\alpha, \beta, |\beta| < \frac{\pi}{2}$ and $-\frac{1}{2} \leq \gamma < 1$. Then

$$f(\mathrm{z}) \in \mathcal{B}_m(\alpha,\beta,\gamma) \;\text{ iff }\; \tilde{\mathcal{J}}\left(\alpha,\beta,\gamma,f\right) \in \mathcal{P}_m \text{ for } \mathrm{z} \in \mathbb{E}, \quad m \geq 2.$$

For any real α, $-\frac{1}{2} \leq \gamma < 1$, $\beta = 0$, we note that the identity function belongs to $\mathcal{B}_m(\alpha,0,\gamma)$ so that $\mathcal{B}_m(\alpha,\beta,\gamma)$ is not empty in general.

Special Cases.

(**i**) For $\beta = 0$, we have the class $\mathcal{B}_m(\alpha,\gamma)$, introduced and studied in [81].

(ii) For $m = 2$, $0 \leq \alpha \leq 1$ and $\beta = \gamma = 0$, we have $\mathcal{B}_2(\alpha,0,0)$ is a subclass of $\mathcal{A}$ introduced by Mocanu [61].

(iii) For $m = 2$ and $\beta = 0$, we have the class $\mathcal{B}_2(\alpha,0,\gamma)$ which consists entirely of univalent functions,see [123].

(**iv**) For $m = 2$ and $\gamma = 0$, we have the class $\mathcal{B}_2(\alpha,\beta,0) = \mathcal{SC}\left(\alpha,\beta\right)$, introduced in [124].

(**v**) For $\alpha = \beta = 0$, we have the class $\mathcal{B}_m(0,0,\gamma) = \mathcal{R}_m$, where $\mathcal{R}_m$ denotes the class of bounded radius rotation, see [38].

(**vi**) For $m = 2$ and $\alpha = \gamma = 0$, we have $\mathcal{B}_2(0,\beta,0) = \mathcal{S}^{\beta}$, the class of spirallike functions defined in [118].

(vii) For $m = 2$, $\alpha = 1$ and $\gamma = 0$, we have $\mathcal{B}_2(1,\beta,0) = \mathcal{SC}_{\beta}$, introduced in [128].

5.2 Main Results

Theorem 5.2.1. Let $\alpha > 0, |\beta| < \frac{\pi}{2}$, $m \geq 2$. Then $\mathcal{M}_m^*(\alpha,\beta) \subset \mathcal{R}_m^*(\beta)$.

Proof. Let $f \in \mathcal{M}_m^*(\alpha,\beta)$ and let

$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} = (\cos\beta)\,h(\mathrm{z}) + i\sin\beta$$

After simple computation, we have

$$\begin{aligned}\mathcal{J}(\alpha,\beta,f(\mathrm{z})) &= \left(e^{i\beta} - \alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} + \alpha\cos\beta\left(1 + \frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\\ &= (\cos\beta)h(\mathrm{z}) + i\sin\beta + \alpha\cos^2\beta\frac{\mathrm{z}h'(\mathrm{z})}{(\cos\beta)h(\mathrm{z}) + i\sin\beta}, \qquad (5.2.1)\end{aligned}$$

and $\mathcal{J}(\alpha,\beta,f(\mathrm{z})) \in \mathcal{P}_m$ in $\mathbb{E}$.

Following the same technique used in [81], we note that there exists an analytic function $\Phi_{\alpha,\beta}(\mathrm{z})$ such that

$$\left(h(\mathrm{z}) * \frac{\Phi_{\alpha,\beta}(\mathrm{z})}{\mathrm{z}}\right) = h(\mathrm{z}) + \frac{\alpha\mathrm{z}h'(\mathrm{z})}{h(\mathrm{z}) + i\tan\beta}. \qquad (5.2.2)$$

where $\alpha > 0$ and $|\beta| < \frac{\pi}{2}$. Now, from (2.7.1), (5.2.1) and (5.2.2), we have

$$(\cos\beta)\,\mathfrak{Re}\left[h_j(\mathrm{z}) + \frac{\alpha\mathrm{z}h_j'(\mathrm{z})}{h_j(\mathrm{z}) + i\tan\beta}\right] > 0, \quad \text{for } j = 1,2 \quad \text{and} \quad \mathrm{z} \in \mathbb{E}.$$

We formulate the functional $\Psi(u,v)$ by choosing $u = h_j(\mathrm{z})$ and $v = \mathrm{z}p_j'(\mathrm{z})$ as

$$\Psi(u,v) = u + \frac{\alpha v}{u + i\tan\beta}.$$

We verify the conditions of Lemma 2.10.3 as follows.

i) $\Psi(u,v)$ is continuous in a domain $\mathfrak{D} \setminus \{u \neq -i\tan\beta\} \subset \mathbb{C}^2$.

ii) $(1,0) \in \mathfrak{D}$ and $\Psi(1,0) = 1 > 0$.

iii) Replacing u by iu_2 and v by v_1 in $\Psi(u,v)$ and taking real of that, we have

$$\begin{aligned}\mathfrak{Re}\Psi(iu_2,v_1) &= \mathfrak{Re}\left\{\frac{\alpha v_1}{i(u_2+\tan\beta)}\right\}\\ &\leq -\mathfrak{Re}\left\{\frac{\alpha(1+u_2^2)}{2i(u_2+\tan\beta)}\right\}\\ &= \mathfrak{Re}\left\{\frac{i\alpha(1+u_2^2)}{2(u_2+\tan\beta)}\right\}=0.\end{aligned}$$

Where we have used $v_1 \leq -\frac{(1+u_2^2)}{2}$. This shows that all the conditions of Lemma 2.10.3 are satisfied and therefore $h_j(\mathrm{z}) \in \mathcal{P}$, $\mathrm{z} \in \mathbb{E}$, $j=1,2$. Consequently $h \in \mathcal{P}_m$ in $\mathbb{E}$ and this completes the proof. $\square$

Theorem 5.2.2. For $0 \leq \alpha_1 < \alpha_2$,

$$\mathcal{M}_m^*(\alpha_2,\beta) \subset \mathcal{M}_m^*(\alpha_1,\beta).$$

Proof. Let $f \in \mathcal{M}_m^*(\alpha_2,\beta)$. Then

$$\begin{aligned}\mathcal{J}(\alpha_1,\beta,f(\mathrm{z})) &= \left(1-\frac{\alpha_1}{\alpha_2}\right)e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\\ &\quad+\frac{\alpha_1}{\alpha_2}\left[\left(e^{i\beta}-\alpha_2\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\alpha_2\cos\beta\left(1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right]\end{aligned}$$

$$=\left(1-\frac{\alpha_1}{\alpha_2}\right)h_1(\mathrm{z})+\frac{\alpha_1}{\alpha_2}h_2(\mathrm{z})=H(\mathrm{z}),$$

where

$$h_1(\mathrm{z}) = e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} \in \mathcal{P}_m, \quad \text{by Theorem 5.2.1}$$

$$h_2(\mathrm{z}) = \mathcal{J}(\alpha_2,\beta,f(\mathrm{z})) \in \mathcal{P}_m, \quad \text{since } f(\mathrm{z}) \in \mathcal{M}_m^*(\alpha_2,\beta).$$

The class $\mathcal{P}_m$ is known [13] to be a convex set, and hence it follows that $H \in \mathcal{P}_m$. This implies that $f \in \mathcal{M}_m^*(\alpha_1,\beta)$. $\square$

We now deal with the converse case of Theorem 5.2.1 as follows.

Theorem 5.2.3. Let $f \in \mathcal{R}_m^*(\beta)$. Then, for $\alpha > 0$, $f \in \mathcal{M}_m^*(\alpha, \beta)$ for $|\mathrm{z}| < \mathrm{r}_0$, where

$$\mathrm{r}_0 = \frac{\sec\beta}{\sqrt{A+\sqrt{A^2-\sec^4\beta}}}, \quad A = 2(\alpha+1)^2 + \tan^2\beta - 1. \tag{5.2.3}$$

This result is sharp.

Proof. Let

$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} = (\cos\beta)\, h(\mathrm{z}) + i\sin\beta, \quad h(\mathrm{z}) \in \mathcal{P}_m$$

where h is given by (2.7.1).

Proceeding as in Theorem 5.2.1, we have

$$\begin{aligned} \mathcal{J}(\alpha, \beta, f(\mathrm{z})) &= \left(\frac{m}{4}+\frac{1}{2}\right)\left[(\cos\beta)\, h_1(\mathrm{z}) + i\sin\beta + \left(\alpha\cos^2\beta\right)\frac{\mathrm{z}h_1'(\mathrm{z})}{(\cos\beta)\, h_1(\mathrm{z}) + i\sin\beta}\right] \\ &\quad - \left(\frac{m}{4}-\frac{1}{2}\right)\left[(\cos\beta)\, h_2(\mathrm{z}) + i\sin\beta + \left(\alpha\cos^2\beta\right)\frac{\mathrm{z}h_2'(\mathrm{z})}{(\cos\beta)\, h_2(\mathrm{z}) + i\sin\beta}\right] \end{aligned} \tag{5.2.4}$$

Now, for $j = 1, 2$

$$\begin{aligned} &\mathfrak{Re}\left[(\cos\beta)\, h_j(\mathrm{z}) + i\sin\beta + \left(\alpha\cos^2\beta\right)\frac{\mathrm{z}h_j'(\mathrm{z})}{(\cos\beta)\, h_j(\mathrm{z}) + i\sin\beta}\right] \\ &= \cos\beta\,\mathfrak{Re}\left[h_j(\mathrm{z}) + \frac{\alpha\mathrm{z}h_j'(\mathrm{z})}{h_j(\mathrm{z}) + i\tan\beta}\right]. \end{aligned}$$

Using Lemma 2.10.1, with $\mu_1 = \alpha > 0$, $\mu_2 = i\tan\beta$, it follows that

$$\mathfrak{Re}\left[h_j(\mathrm{z}) + \frac{\alpha\mathrm{z}h_j'(\mathrm{z})}{h_j(\mathrm{z}) + i\tan\beta}\right] > 0, \quad \text{for } \mathrm{z} < \mathrm{r}_0,$$

where r_0 is given by (5.2.3). Consequently, from (5.2.4), it follows that $\mathrm{J}(\alpha, \beta, f) \in \mathcal{P}_m$ for $|\mathrm{z}| < \mathrm{r}_0$.

Theorem 5.2.4. Let $f \in \mathcal{R}_m^*(\beta)$. Then $f \in \mathcal{R}_m$ for $|\mathrm{z}| < \mathrm{r}_\beta$, where

$$\mathrm{r}_\beta = \frac{1}{\cos\beta + \sin\beta}. \tag{5.2.5}$$

This result is sharp.

Proof. Let
$$\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}=h(\mathrm{z})=\left(\frac{m}{4}+\frac{1}{2}\right)h_1(\mathrm{z})-\left(\frac{m}{4}-\frac{1}{2}\right)h_2(\mathrm{z}).$$
Then
$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}=\left(\frac{m}{4}+\frac{1}{2}\right)\left(e^{i\beta}h_1(\mathrm{z})\right)-\left(\frac{m}{4}-\frac{1}{2}\right)\left(e^{i\beta}h_2(\mathrm{z})\right).$$
Since $f\in\mathcal{R}_m^*(\beta)$, it follows that $e^{i\beta}h_j(\mathrm{z})\in\mathcal{P}_m$, $j=1,2$. Using Lemma 2.10.4, we have

$$\mathfrak{Re}h_j(\mathrm{z})\geq\frac{1-2(\cos\beta)\mathrm{r}+(\cos 2\beta)\mathrm{r}^2}{1-\mathrm{r}^2},$$

and thus it follows that $h_j(\mathrm{z})\in\mathcal{P}$ for $|\mathrm{z}|<\mathrm{r}_\beta$, where r_β is given by $(5.2.5)$.

The function
$$h(\mathrm{z})=\left(\frac{m}{4}+\frac{1}{2}\right)\frac{1+\mathrm{z}}{1-\mathrm{z}}-\left(\frac{m}{4}-\frac{1}{2}\right)\frac{1-\mathrm{z}}{1+\mathrm{z}}$$
gives us the sharpness. □

Theorem 5.2.5. Let $f\in\mathcal{A}$. Then $f\in\mathrm{B}_m(\alpha,\beta,\gamma)$, $\alpha\neq 0$, iff, there exists a function $g\in\mathcal{B}_m(0,\beta,\gamma)=\mathcal{R}_m^*(\beta)$ such that

$$f(\mathrm{z})=\left[\varsigma\int_0^{\mathrm{z}}t^{\varsigma-1}\left(\frac{g(t)}{t}\right)^{\frac{(1-\gamma)e^{i\beta}}{\alpha\cos\beta}}dt\right]^{\frac{1}{\varsigma}},\tag{5.2.6}$$

where
$$\varsigma=1+\frac{(1-\gamma)(e^{i\beta}-\alpha\cos\beta)}{\alpha\cos\beta}.$$

Proof. From $(5.2.6)$, we have after some computation,

$$\begin{aligned}e^{i\beta}\frac{\mathrm{z}g'(\mathrm{z})}{g(\mathrm{z})}&=\left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\frac{\alpha\cos\beta}{1-\gamma}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\\&=\tilde{\mathcal{J}}(\alpha,\beta,\gamma,f(\mathrm{z})).\end{aligned}$$

If the right hand side belongs to $\mathcal{P}_m$, so does the left hand side and conversely.

Theorem 5.2.6. Let $f \in \mathcal{B}_m(\alpha,\beta,\gamma)$. Then the function $g \in \mathcal{R}_m^*(\beta)$, where

$$\left(\frac{g(\mathrm{z})}{\mathrm{z}}\right)^{e^{i\beta}} = \left(\frac{f(\mathrm{z})}{\mathrm{z}}\right)^{e^{i\beta}-\alpha\cos\beta} \left(f'(\mathrm{z})\right)^{\frac{\alpha\cos\beta}{1-\gamma}}. \tag{5.2.7}$$

Proof. Logarithmic differentiation of (5.2.7) and simple calculations yield

$$e^{i\beta}\frac{\mathrm{z}g'(\mathrm{z})}{g(\mathrm{z})} = \left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\frac{\alpha\cos\beta}{1-\gamma}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right),$$

and since $f \in \mathcal{B}_m(\alpha,\beta,\gamma)$, we immediately obtain the required result. □

Theorem 5.2.7. $\mathcal{B}_m(\alpha,\beta,\gamma) \subset \mathcal{B}_m(\alpha_1,\beta,\gamma)$, $\quad 0 \leq \alpha_1 < \alpha$.

Proof. Let $f \in \mathcal{B}_m(\alpha,\beta,\gamma)$. Now

$$\begin{aligned} & \frac{1}{1-\gamma}\left[\left(e^{i\beta}-\alpha_1\cos\beta\right)(1-\gamma)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\alpha_1\cos\beta\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right] \\ = & \frac{\alpha_1}{\alpha}\left[\left(e^{i\beta}-\alpha_2\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\frac{\alpha_2\cos\beta}{(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right]-\left(\frac{\alpha_1-\alpha}{\alpha}\right)e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} \\ = & \frac{\alpha_1}{\alpha}h_3(\mathrm{z})+\left(1-\frac{\alpha_1}{\alpha}\right)h_4(\mathrm{z}) = H(\mathrm{z}), \end{aligned}$$

where

$$h_3(\mathrm{z}) = \left\{\left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\frac{\alpha\cos\beta}{(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right\} \in \mathcal{P}_m$$

$$h_4(\mathrm{z}) = e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} \in \mathcal{P}_m, \qquad \text{(by Theorem 5.2.1)}.$$

Since $\mathcal{P}$ is a convex set, $H \in \mathcal{P}_m$ and this completes the proof. □

Theorem 5.2.8. Let $f \in \mathcal{B}_m(\alpha,\beta,\gamma)$, $\alpha > 0$. Then f is univalent in $\mathbb{E}$ for

$$m < \frac{2\left[\alpha\cos\beta\,(1+2\gamma)+1-\gamma\right]}{1-\gamma}.$$

Proof. Let $f \in \mathcal{B}_m(\alpha,\beta,\gamma)$. Then

$$H(\mathrm{z})=\left\{\left(e^{i\beta}-\alpha\cos\beta\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\frac{\alpha\cos\beta}{(1-\gamma)}\left(1-\gamma+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right)\right\}\in\mathcal{P}_m,\quad \mathrm{z}\in\mathbb{E}.$$

That is

$$\begin{aligned}&\left[\frac{(1-\gamma)\cos\beta+\alpha\gamma\cos\beta}{\alpha\cos\beta}-1\right]\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}+\frac{\left(\mathrm{z}f'(\mathrm{z})\right)'}{f'(\mathrm{z})}+i\left[\frac{(1-\gamma\sin\beta)}{\alpha\cos\beta}\right]\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\\ =\;& H(\mathrm{z})+\gamma.\end{aligned}$$

Therefore, for $0\le\theta_1<\theta_2\le 2\pi$, $\mathrm{z}=\mathrm{r}e^{i\theta}$, we have

$$\begin{aligned}&\int_{\theta_1}^{\theta_2}\left\{\mathfrak{Re}\left[1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}+(\beta_1-1)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right]-\alpha_{1\,\mathrm{Im}}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right\}d\theta\\ >\;& -\left[\left(\frac{m}{2}-1\right)\left(\frac{1-\gamma}{\alpha\cos\beta}\right)\pi-2\gamma\pi\right],\end{aligned}$$

where

$$\begin{aligned}\beta_1&=\frac{(1-\gamma)\cos\beta+\alpha\gamma\cos\beta}{\alpha\cos\beta}=\frac{1-\gamma+\alpha\gamma}{\alpha},\\ \alpha_1&=\frac{1-\gamma}{\alpha}\tan\beta.\end{aligned}$$

Using Lemma 2.10.2. it follows that f is univalent if

$$\left[\left(\frac{m}{2}-1\right)\left(\frac{1-\gamma}{\alpha\cos\beta}\right)-2\gamma\right]\le 1. \qquad \square$$

Theorem 5.2.9. Let $f\in\mathcal{B}_m(\alpha,\beta,\gamma)$, $\alpha>0$ and let $L_{\mathrm{r}}(f)$ denote the length of the curve $\mathcal{C}$,

$$C=f\left(\mathrm{r}e^{i\theta}\right),\quad 0\le\theta\le 2\pi,\quad\text{and}\quad M(\mathrm{r})=\max_{0\le\theta\le 2\pi}\left|f\left(\mathrm{r}e^{i\theta}\right)\right|.$$

Then, for $0 < \mathrm{r} < 1$,

$$L_{\mathrm{r}}(f) \leq \frac{\pi M(\mathrm{r})}{\alpha \cos\beta} \begin{cases} \left[m\left(1+\left|\alpha\cos\beta - e^{i\beta}\right|\right) + \frac{2\alpha\gamma\cos\beta}{1-\gamma}\right], & 0 < \alpha < 2, \\ m\left(1+\sqrt{\alpha(\alpha-2)\cos^2\beta+1}\right) + \frac{2\alpha\gamma\cos\beta}{1-\gamma}, & \alpha \geq 2. \end{cases}$$

Proof. With $\mathrm{z} = \mathrm{r}e^{i\theta}$, and integration by parts, we have

$$\begin{aligned} L_{\mathrm{r}}(f) &= \int_0^{2\pi} \left|\mathrm{z}f'(\mathrm{z})\right| d\theta \\ &= \int_0^{2\pi} \mathrm{z}f'(\mathrm{z})e^{-i\arg\left(\mathrm{z}f'(\mathrm{z})\right)} d\theta \\ &= \int_0^{2\pi} f(\mathrm{z})e^{-i\arg\left(\mathrm{z}f'(\mathrm{z})\right)} \mathfrak{Re}\left\{\frac{\left(\mathrm{z}f'(\mathrm{z})\right)'}{f'(\mathrm{z})}\right\} d\theta \\ &\leq \frac{M(\mathrm{r})}{\alpha\cos\beta}\int_0^{2\pi} \left|\mathfrak{Re}\tilde{\mathcal{J}}(\alpha,\beta,\gamma,f(\mathrm{z})) + \left(\alpha\cos\beta - e^{i\beta}\right)\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} + \frac{\alpha\gamma\cos\beta}{1-\gamma}\right| d\theta \\ &\leq \frac{M(\mathrm{r})}{\alpha\cos\beta}\int_0^{2\pi} \left|\mathfrak{Re}\tilde{\mathcal{J}}(\alpha,\beta,\gamma,f(\mathrm{z}))\, d\theta\right. \\ &\quad + \frac{M(\mathrm{r})}{\alpha\cos\beta}\left[\left|\alpha\cos\beta - e^{i\beta}\right| \int_0^{2\pi}\left|\mathfrak{Re}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right| d\theta + \int_0^{2\pi}\left|\mathfrak{Re}\frac{\alpha\gamma\cos\beta}{1-\gamma}\right| d\theta\right] \\ &\leq \frac{M(\mathrm{r})}{\alpha\cos\beta}\left[m\pi + \left|\alpha\cos\beta - e^{i\beta}\right|(m\pi) + \frac{2\alpha\gamma\cos\beta}{1-\gamma}\pi\right], \end{aligned}$$

and this gives us the required result. □

Remark 5.2.1. For $\alpha > 0$ and $f \in \mathcal{B}_m(\alpha,\beta,\gamma)$, we can write Theorem 5.2.9 as.

$$L_{\mathrm{r}}(f) \leq \frac{\pi M(\mathrm{r})}{\alpha\cos\beta}\left\{m(2+\alpha\cos) + \frac{2\gamma\cos\beta}{1-\gamma}\right\}.$$

Theorem 5.2.10. Let $f \in \mathcal{B}_m(\alpha,\beta,\gamma)$, $\alpha > 0$ and be given by (2.1.1). Then, for $\ell \geq 2$,

$$\ell\left|a_\ell\right| = O(1)\, M\left(\frac{\ell-1}{\ell},\right)$$

Where $O(1)$ is constant depending on the parameters α, β, γ and m only.

Proof. The result follows immediately from Theorem 5.2.9, since

$$\ell\left|a_{\ell}\right| \leq \frac{1}{2\pi \mathrm{r}^{\ell}} \int_{0}^{2\pi} \left|\mathrm{z} f^{'}\left(\mathrm{z}\right)\right| d\theta = \frac{1}{2\pi \mathrm{r}^{\ell}} L_{\mathrm{r}}\left(f\right). \qquad \square$$

Theorem 5.2.11. The class $\mathcal{B}_{m}(0,\beta,\gamma)$ is preserved under the integral operator $\mathfrak{I}$ defined as.

$$\mathfrak{I}(f) = \frac{\eta+1}{\mathrm{z}^{\eta}} \int_{0}^{\mathrm{z}} t^{\eta-1} f(t) dt, \quad \eta > -1. \tag{5.2.8}$$

Proof. Set

$$\begin{aligned} e^{i\beta} \frac{\mathrm{z}\mathfrak{I}^{'}(\mathrm{z})}{p\mathfrak{I}(\mathrm{z})} &= (\cos\beta) h(\mathrm{z}) + i\sin\beta \\ &= \left[\left(\frac{m}{4}+\frac{1}{2}\right)(\cos\beta)\, h_{1}\left(\mathrm{z}\right) + i\sin\beta\right] \\ &\quad - \left[\left(\frac{m}{4}-\frac{1}{2}\right)(\cos\beta)\, h_{2}\left(\mathrm{z}\right) + i\sin\beta\right], \end{aligned} \tag{5.2.9}$$

Then, from $(5.2.8)$, we have

$$e^{i\beta} \frac{\mathrm{z} f^{'}(\mathrm{z})}{f(\mathrm{z})} = \cos\beta \left[h(\mathrm{z}) + i\sin\beta + \frac{\mathrm{z} h^{'}(\mathrm{z})}{h(\mathrm{z}) + \eta\sec\beta + i\tan\beta}\right]. \tag{5.2.10}$$

Following the same technique used in [81], we note that there exists an analytic function $\Phi_{\eta,\beta}\left(\mathrm{z}\right)$ such that

$$\left(h(\mathrm{z}) * \frac{\Phi_{\eta,\beta}\left(\mathrm{z}\right)}{\mathrm{z}}\right) = \left[h\left(\mathrm{z}\right) + \frac{\mathrm{z} h^{'}\left(\mathrm{z}\right)}{h\left(\mathrm{z}\right) + \eta\sec\beta + i\tan\beta}\right], \tag{5.2.11}$$

where $\eta > -1$ and $|\beta| < \frac{\pi}{2}$. From $(5.2.9)$, $(5.2.10)$ and $(5.2.11)$, we have

$$\begin{aligned} e^{i\beta} \frac{\mathrm{z} f^{'}(\mathrm{z})}{f(\mathrm{z})} &= \left(\frac{m}{4}+\frac{1}{2}\right)\left[(\cos\beta)\, h_{1}\left(\mathrm{z}\right) + i\sin\beta + \frac{\left(\cos^{2}\beta\right) \mathrm{z} h_{1}^{'}\left(\mathrm{z}\right)}{(\cos\beta)\, h_{1}\left(\mathrm{z}\right) + \eta + i\sin\beta}\right] \\ &\quad - \left(\frac{m}{4}-\frac{1}{2}\right)\left[(\cos\beta)\, h_{2}\left(\mathrm{z}\right) + i\sin\beta + \frac{\left(\cos^{2}\beta\right) \mathrm{z} h_{2}^{'}\left(\mathrm{z}\right)}{(\cos\beta)\, h_{2}\left(\mathrm{z}\right) + \eta + i\sin\beta}\right]. \end{aligned}$$

Since

$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\in\mathcal{P}_m,\quad \text{for z}\in\mathbb{E},$$

it follows that

$$\mathfrak{Re}\left[(\cos\beta)\,h_j(\mathrm{z})+i\sin\beta+\frac{(\cos^2\beta)\,\mathrm{z}h_j'(\mathrm{z})}{(\cos\beta)\,h_j(\mathrm{z})+\eta+i\sin\beta}\right]>0,\quad \mathrm{z}\in\mathbb{E},\quad j=1,2.$$

We want to show that $\mathfrak{Re}h_j(\mathrm{z})>0$ in $\mathbb{E}$, which will imply that $h\in\mathcal{P}_m$ in $\mathbb{E}$. We proceed by performing the functional $\Psi(u,v)$ with $u=h_j(\mathrm{z})$ and $v=\mathrm{z}h'(\mathrm{z})$. Thus

$$\Psi(u,v)=u+i\tan\beta+\frac{v}{u+\eta\sec\beta+i\tan\beta}.$$

We note that the first two conditions of Lemma 2.10.3 are clearly satisfied. We verify the third condition as follows.

$$\begin{aligned}\mathfrak{Re}\Psi(iu_2,v_1) &= \mathfrak{Re}\left[\frac{v_1}{\eta\sec\beta+i(u_2+\tan\beta)}\right]\\ &= \frac{\eta(\sec\beta)\,v_1}{\eta^2\sec^2\beta+(u_2+\tan\beta)^2}\\ &\leq -\frac{1}{2}\frac{\eta\sec\beta\,(1+u_2^2)}{\eta^2\sec^2\beta+(u_2+\tan\beta)^2}\leq 0.\end{aligned}$$

Where we have used $v_1\leq-\frac{(1+u_2^2)}{2}$. This proves $\mathfrak{Re}(h_j(\mathrm{z}))>0,\ \ j=1,2.$ □

The following result is about the converse of Theorem 5.2.11 as follows.

Theorem 5.2.12. Let $\mathfrak{I}\in\mathcal{B}_m(0,\beta,\gamma)$ and be defined by (5.2.8). Then $f\in\mathcal{B}_m(0,\beta,\gamma)$ for $|\mathrm{z}|<\mathrm{r}_1$, where the value of r_1 is exact and is given by (2.10.1) in Lemma 2.10.1 with $\mu_1=1$ and $\mu_2=(\eta\sec\beta+i\tan\beta)$.

5.4 Conclusion

We have used the idea of bounded Mocanu variation to introduce a new class of analytic functions, defined in the open unit disk $\mathbb{E}$. This class unifies a number of classes previously studied such as those of functions with bounded radius rotation and bounded Mocanu [61] variation. It also generalizes the concept of β-spiral likeness in some sense. Some interesting properties of this class including inclusion results, arclength problems and a sufficient condition for univalency have been studied.

Chapter 6

Coefficient Bounds for a Subclass of Multivalent Functions of Reciprocal Order

6.1 Introduction

Let $N(\lambda)$ and $M(\lambda)$ denote the usual classes of starlike and convex functions of reciprocal order λ, $\lambda > 1$, and are defined by

$$\begin{aligned} N(\lambda) &= \left\{ f \in \mathcal{A} : \mathfrak{Re}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} < \lambda, \ \ (\mathrm{z} \in \mathbb{E}) \right\}, \\ M(\lambda) &= \left\{ f \in \mathcal{A} : 1+ \mathfrak{Re}\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})} < \lambda, \ \ (\mathrm{z} \in \mathbb{E}) \right\}. \end{aligned}$$

These classes were introduced by Uralegaddi et.al. [125] in 1994 and then studied by the authors in [88].

In 2002, Owa and Srivastava [89] generalized this idea for the classes of multivalent starlike and multivalent convex functions of reciprocal order λ with $\lambda > p$, and further investigated by Polatoglu et.al. [96]. Recently in 2011, Uyanik et.al. [126] extended this idea to the classes of p-valently spirallike and p-valently Robertson functions and discussed coefficient inequalities and sufficient conditions for the functions of these classes. We generalize and define the following new class of analytic spirallike multivalent functions with reciprocal order.

Definition 6.1.1 An analytic multivalent function $f \in \mathrm{S}\mathcal{C}_p^{\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \lambda)$, iff

$$\mathfrak{Re}\left\{ \frac{2e^{i\beta}}{\mathfrak{b}} \left(\frac{\mathbb{k}_p(\mathfrak{a}+1, \mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a}, \mathfrak{c})f(\mathrm{z})} - 1 \right) \right\} < (\lambda - 1)\cos\beta,$$

where $\mathfrak{b} \in \mathbb{C}\diagdown\{0\}$, β is real with $|\beta| < \frac{\pi}{2}$, $\lambda > 1$.

Special Cases

i) Let $\beta = 0$, $\mathfrak{a} = 1 = \mathfrak{c}$ and $\mathfrak{b} = 2$, we obtain $\mathrm{S}\mathcal{C}_p^0(1, 2, 1, \lambda) = M_p(\lambda)$, studied in [88] and further for $p = 1$, we have the class $M(\lambda)$, introduced and studied in [63, 125].

ii) Let $p = \mathfrak{c} = \mathfrak{b} = 1$, $\beta = 0$ and $\mathfrak{a} = 2$, we have $\mathcal{SC}_1^0(1, 2, 1, \lambda) = N(\lambda)$, studied in

[63, 125].

iii) Let $\mathfrak{a} = 1 = \mathfrak{c}$ and $\mathfrak{b} = 2$, we get the class $\mathcal{SC}_p^{\beta}(1,2,1,\lambda) = \mathcal{S}_p(\beta,\lambda)$ and for $\mathfrak{a} = 2$, $\mathfrak{b} = \mathfrak{c} = 1$, we obtain $\mathcal{SC}_p^{\beta}(2,1,1,\lambda) = \mathcal{C}_p(\beta,\lambda)$, introduced and studied in [126].

iv) Let $\beta = 0$, $\mathfrak{a} = 2$, $\mathfrak{b} = p = 1$, and $\mathfrak{c} = 2-\lambda$, we obtain the class $\mathcal{SC}_1^{0}(2,1,2-\lambda,\lambda) = \mathcal{P}_{\alpha}(\lambda)$, see [16].

2.1 Preliminary Results

Lemma 6.2.1. A function f given in (2.4.1) belongs to the class $\mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$, iff

$$e^{i\beta}\left(1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}\right) \prec q(\mathrm{z}), \quad \mathrm{z} \in \mathbb{E},$$

where

$$q(\mathrm{z}) = \frac{\cos\beta - \{(2\lambda-1)\cos\beta + i\sin\beta\}\,\mathrm{z}}{1-\mathrm{z}}.$$

for some real $\beta(|\beta| < \frac{\pi}{2})$ and $\lambda > p$.

The proof of above Lemma is similar to that of Theorem 1 in [126] so we omit the proof.

6.3 Main Results

Theorem 6.3.1. If $f \in \mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$, then

$$|a_2| \leq \frac{|\mathfrak{b}|\,|\eta|\,\mathfrak{c}}{p},$$

and

$$|a_{\ell+p-1}| \leq \frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(\ell+p-2)\phi_{\ell-1}(\mathfrak{a})}\prod_{j=1}^{\ell-2}\left(1+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+j)}\right), \quad \ell \geq 3, \tag{6.3.1}$$

where $\phi_{\ell-1}(\delta)$ is given by (2.9.2) and

$$\eta = (1-\lambda)\cos\beta + i\sin\beta. \tag{6.3.2}$$

Proof. Let $f \in \mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. Then we have

$$\mathfrak{Re}\left\{\frac{2e^{i\beta}}{\mathfrak{b}}\left(\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}-1\right)\right\}<(\lambda-1)\cos\beta,\quad \mathrm{z}\in\mathbb{E}.$$

Now let us define a function h by

$$e^{i\beta}\left(1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}\right)=((1-\lambda)h(\mathrm{z})+\lambda)\cos\beta+i\sin\beta. \tag{6.3.3}$$

Then (6.3.3) can be written as

$$1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}=1+\frac{(1-\lambda)\cos\beta+i\sin\beta}{e^{i\beta}}\sum_{\ell=1}^{\infty}d_\ell \mathrm{z}^\ell,$$

or equivalently

$$2e^{i\beta}\left(\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})-\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})\right)=\mathfrak{b}\eta\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})\sum_{\ell=1}^{\infty}d_\ell \mathrm{z}^\ell, \tag{6.3.4}$$

where η is given by (6.3.2). From (6.3.4) and (2.9.3) we have

$$2e^{i\beta}\left[\mathrm{z}\left(\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})\right)'-p\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})\right]=\mathfrak{a}\mathfrak{b}\eta\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})\sum_{\ell=1}^{\infty}d_\ell \mathrm{z}^\ell,$$

that is,

$$2e^{i\beta}\left[\sum_{\ell=1}^{\infty}(\ell+p-1)\phi_\ell(\mathfrak{a})a_{\ell+p}\mathrm{z}^{\ell+p}\right]=\mathfrak{a}\mathfrak{b}\eta\left[\mathrm{z}^p+\sum_{\ell=1}^{\infty}\phi_\ell(\mathfrak{a})a_{\ell+p}\mathrm{z}^{\ell+p}\right]\left(\sum_{\ell=1}^{\infty}d_\ell \mathrm{z}^\ell\right).$$

Comparing the coefficients of $\mathrm{z}^{\ell+p-1}$ on both sides, we obtain

$$2e^{i\beta}(\ell+p-2)\phi_{\ell-1}(\mathfrak{a})a_{\ell+p-1}=\mathfrak{a}\mathfrak{b}\eta\left\{d_1a_{\ell-2}\phi_{\ell+p-2}(\mathfrak{a})+...+d_{\ell-1}\right\}. \tag{6.3.5}$$

Taking absolute on both sides and then applying Lemma 2.10.8, we have

$$|a_{\ell+p-1}| \leq \frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(\ell+p-2)\phi_{\ell-1}(\mathfrak{a})}\left\{1+\phi_1(\mathfrak{a})\,|a_{p+1}|+\ldots+\phi_{\ell-2}(\mathfrak{a})\,|a_{\ell+p-2}|\right\}. \tag{6.3.6}$$

We now apply mathematical induction on (6.3.6). So for $\ell = 2$

$$|a_{p+1}| \leq \frac{|\mathfrak{b}|\,|\eta|\,\mathfrak{c}}{p}.$$

which shows that the result is true for $\ell = 2$. For $\ell = 3$

$$|a_{p+2}| \leq \frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+1)\phi_2(\mathfrak{a})}\left\{1+\phi_1(\mathfrak{a})\,|a_{p+1}|\right\}, \tag{6.3.7}$$

and using the bound of $|a_{p+1}|$ in (6.3.7), we have

$$|a_{p+2}| \leq \frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+1)\phi_2(\mathfrak{a})}\left\{1+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{p}\right\}.$$

Therefore (6.3.1) holds for $\ell = 3$.

Assume that (6.3.1) is true for $\ell = i$, that is,

$$|a_{i+p-1}| \leq \frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(i+p-2)\phi_{i-1}(\mathfrak{a})}\prod_{j=1}^{i-2}\left(1+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+j)}\right).$$

Consider

$$\begin{aligned}
|a_{i+p}| \;&\leq\; \frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(i+p-1)\phi_i(\mathfrak{a})}\left\{\left(1+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{p}\right)+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+1)}\left(1+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{p}\right)\right.\\
&\qquad\left.+\ldots+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+i-1)}\prod_{j=1}^{i-2}\left(1+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+j)}\right)\right\}\\
&=\; \frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(i+p-1)\phi_i(\mathfrak{a})}\prod_{j=1}^{i-1}\left(1+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+j)}\right).
\end{aligned}$$

Therefore the result is true for $\ell = i+1$ and hence by using mathematical induction, (6.3.1) holds true for all $\ell \geq 3$. $\square$

The following corollaries which were proved by Owa and Nishawski [89] comes as a special case from Theorem 2.3.1 by varying the parameter $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, p$ and β.

Corollary 6.3.1. If $f \in M(\lambda)$, then

$$|a_\ell| \leq \prod_{j=2}^{\ell} \frac{(j+2\lambda-4)}{(\ell-1)!}, \quad \text{for all } \ell \geq 2.$$

Corollary 6.3.2. If $f \in N(\lambda)$, then

$$|a_\ell| \leq \prod_{j=2}^{\ell} \frac{(j+2\lambda-4)}{\ell!}, \quad \text{for all } \ell \geq 2.$$

Theorem 6.3.2. Let $f \in$A(p) satisfy

$$\sum_{\ell=1}^{\infty} \left| \left(1+\frac{2\ell}{\mathfrak{a}\mathfrak{b}}\right) e^{i\beta} - \lambda \cos\beta \right| \phi_\ell(\mathfrak{a})\, |a_{\ell+p}| < \left| \lambda\cos\beta - e^{i\beta} \right|. \tag{6.3.8}$$

Then $f \in \mathcal{SC}_p^\beta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \lambda)$.

Proof. To prove that f belongs to $\mathcal{SC}_p^\beta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \lambda)$, we need to prove that

$$\left| \frac{e^{i\beta}\left(1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(z)}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(z)}\right) - 1}{e^{i\beta}\left(1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(z)}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(z)}\right) - (2\lambda\cos\beta - 1)} \right| < 1. \tag{6.3.9}$$

For this consider the left hand side of (6.3.9), we have

$$\begin{aligned} LHS &= \left| \frac{e^{i\beta}\left(1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(z)}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(z)}\right) - 1}{e^{i\beta}\left(1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(z)}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(z)}\right) - (2\lambda\cos\beta - 1)} \right| \\ &= \left| \frac{\left(e^{i\beta}-1\right)\mathfrak{b}z^p + \sum_{\ell=1}^{\infty}\left(\left(\mathfrak{b}+\frac{2\ell}{\mathfrak{a}}\right)e^{i\beta}-1\right)\phi_\ell(\mathfrak{a})a_{\ell+p}z^{\ell+p}}{\left(\mathfrak{b}e^{i\beta}-2\mathfrak{b}\lambda\cos\beta+\mathfrak{b}\right) + \sum_{\ell=1}^{\infty}\left(\left(\mathfrak{b}+\frac{2\ell}{\mathfrak{a}}\right)e^{i\beta}-2\mathfrak{b}\lambda\cos\beta+\mathfrak{b}\right)\phi_\ell(\mathfrak{a})a_{\ell+p}z^{\ell+p}} \right| \\ &\leq \frac{\left|\left(e^{i\beta}-1\right)\right| + \sum_{\ell=1}^{\infty}\left|\left(1+\frac{2\ell}{\mathfrak{a}\mathfrak{b}}\right)e^{i\beta}-1\right|\phi_\ell(\mathfrak{a})\,|a_{\ell+p}|}{|2\lambda\cos\beta - e^{i\beta}-1| - \sum_{\ell=1}^{\infty}\left|\left(1+\frac{2\ell}{\mathfrak{a}\mathfrak{b}}\right)e^{i\beta}-2\lambda\cos\beta+1\right|\phi_\ell(\mathfrak{a})\,|a_{\ell+p}|}. \end{aligned}$$

The last expression is bounded above by 1 if

$$\left|e^{i\beta}-1\right|+\sum_{\ell=1}^{\infty}\left|\left(1+\frac{2\ell}{\mathfrak{ab}}\right)e^{i\beta}-1\right|\phi_\ell(\mathfrak{a})\,|a_{\ell+p}| \leq \left|2\lambda\cos\beta-e^{i\beta}-1\right|$$

$$-\sum_{\ell=1}^{\infty}\left|\left(1+\frac{2\ell}{\mathfrak{ab}}\right)e^{i\beta}-2\lambda\cos\beta+1\right|\phi_\ell(\mathfrak{a})\,|a_{\ell+p}|$$

which is equivalent to the condition (6.3.8) and so $f\in\mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. □

Corollary 6.3.3. [89]. If $f\in\mathcal{A}$ satisfies

$$\sum_{\ell=2}^{\infty}\left\{(\ell-1)+|\ell-2\lambda+1|\right\}|a_\ell| \leq 2(\lambda-1)$$

for some $\lambda(\lambda>1)$, then $f\in M(\lambda)$.

Corollary 6.3.4. [16]. A function $f\in\mathcal{P}_\lambda(\lambda)$, iff

$$\sum_{\ell=2}^{\infty}\frac{\Gamma(\ell+1)\Gamma(2-\lambda)}{\Gamma(\ell+1-\lambda)}(\ell-\lambda)\,|a_\ell| \leq (\lambda-1).$$

This result is sharp.

Theorem 6.3.3. Let $f\in\mathcal{SC}_p^{0}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$ and be of the form (2.4.1). Then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{|\mathfrak{b}|\,\mathfrak{c}(\mathfrak{c}+1)(1-\lambda)}{(\mathfrak{a}+1)(p+1)}\max\left\{1,|2\upsilon-1|\right\},$$

where

$$\upsilon=\frac{\mu\mathfrak{bc}(\mathfrak{a}+1)(p+1)(1-\lambda)}{2p^2(\mathfrak{c}+1)}-\frac{\mathfrak{ab}(1-\lambda)}{2p}. \tag{6.3.10}$$

Proof. Let $f\in\mathcal{SC}_p^{0}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. Then from (6.3.5) with $\beta=0$, we have

$$\begin{aligned} a_{p+1} &= \frac{\mathfrak{bc}(1-\lambda)}{2p}d_1 \\ a_{p+2} &= \frac{\mathfrak{bc}(\mathfrak{c}+1)(1-\lambda)}{2(\mathfrak{a}+1)(p+1)}\left\{d_2+\frac{\mathfrak{ab}(1-\lambda)}{2p}d_1^2\right\}. \end{aligned}$$

For any complex number μ, we have

$$\begin{aligned} a_{p+2}-\mu a_{p+1}^2 &= \frac{\mathfrak{b}\mathfrak{c}(\mathfrak{c}+1)(1-\lambda)}{2(\mathfrak{a}+1)(p+1)}\left[d_2-\frac{\mathfrak{b}(1-\lambda)}{2p}\left\{\frac{\mu\mathfrak{c}(\mathfrak{a}+1)(p+1)}{p(\mathfrak{c}+1)}-\mathfrak{a}\right\}d_1^2\right] \\ &= \frac{\mathfrak{b}\mathfrak{c}(\mathfrak{c}+1)(1-\lambda)}{2(\mathfrak{a}+1)(p+1)}\left[d_2-\upsilon d_1^2\right], \end{aligned}$$

where υ is given by (6.3.10).

Taking modulus on both sides and applying Lemma 2.10.8, we have

$$\begin{aligned} \left|a_{p+2}-\mu a_{p+1}^2\right| &= \left|\frac{\mathfrak{b}\mathfrak{c}(\mathfrak{c}+1)(1-\lambda)}{2(\mathfrak{a}+1)(p+1)}\right|\left|d_2-\upsilon d_1^2\right| \\ &\le \frac{|\mathfrak{b}|\ \mathfrak{c}(\mathfrak{c}+1)(1-\lambda)}{(\mathfrak{a}+1)(p+1)}\max\{1,|2\upsilon-1|\}. \end{aligned}$$

This proves the required result. □

Now we find the upper bound of the second Hankel determinant for the functions class $\mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$.

Theorem 6.3.4. Let $f\in\mathcal{SC}_p^{0}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. Then

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right|\le\left[\frac{4\,|\mathfrak{b}|\ \mathfrak{c}(\mathfrak{c}+1)(1-\lambda)}{(p+1)(\mathfrak{a}+1)}\right]^2.$$

Proof. Let $f\in\mathcal{SC}_p^{0}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. Then from (6.3.5) we have

$$a_{p+1} = \frac{\mathfrak{a}\mathfrak{b}(1-\lambda)}{2p\phi_1(\mathfrak{a})}d_1. \tag{6.3.11}$$

$$a_{p+2} = \frac{\mathfrak{a}\mathfrak{b}(1-\lambda)}{2(p+1)\phi_2(\mathfrak{a})}\left\{d_2+\frac{\mathfrak{a}\mathfrak{b}(1-\lambda)}{2p}d_1^2\right\}. \tag{6.3.12}$$

$$a_{p+3} = \frac{\mathfrak{a}\mathfrak{b}(1-\lambda)}{2(p+2)\phi_3(\mathfrak{a})}\left\{d_3+\frac{\mathfrak{a}\mathfrak{b}(1-\lambda)}{2(p+1)}d_2^2+\frac{\{\mathfrak{a}\mathfrak{b}(1-\lambda)\}^2}{4p(p+1)}d_1^2d_2+\frac{\mathfrak{a}\mathfrak{b}(1-\lambda)}{2p}d_1^2\right\} \tag{6.3.13}$$

From (6.3.11), (6.3.12) and (6.3.13) , we obtain

$$
\begin{aligned}
\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| = & \frac{\{\mathfrak{ab}(1-\lambda)\}^2}{4p(p+2)\phi_1(\mathfrak{a})\phi_3(\mathfrak{a})} \times \\
& \left\{d_3+\frac{\mathfrak{ab}(1-\lambda)}{2(p+1)}d_2^2+\frac{\{\mathfrak{ab}(1-\lambda)\}^2}{4p(p+1)}d_1^2d_2+\frac{\mathfrak{ab}(1-\lambda)}{2p}d_1^2\right\} \\
& -\frac{\{\mathfrak{ab}(1-\lambda)\}^2}{4(p+1)^2\phi_2^2(\mathfrak{a})}\left\{d_2^2+\frac{\{\mathfrak{ab}(1-\lambda)\}^2}{4p^2}d_1^4+\frac{\mathfrak{ab}(1-\lambda)}{p}d_1^2d_2\right\}.
\end{aligned}
$$

After some simplification, we have

$$
\begin{aligned}
\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| = \frac{|\mathrm{A}|^2}{4}\big[&\mathrm{B}d_1d_3+\mathrm{C}d_1d_2^2 \\
&+\mathrm{E}d_1^3d_2+\mathrm{F}d_1^3-\mathrm{G}d_2^2-\mathrm{H}d_1^4-\mathrm{K}d_1^2d_2\big]\,,
\end{aligned}
\qquad (6.3.14)
$$

where

$$
\begin{array}{lll}
\mathrm{A}=\mathfrak{ab}(1-\lambda), & \mathrm{B}=\frac{1}{p(p+1)\phi_1(\mathfrak{a})\phi_3(\mathfrak{a})}, & \mathrm{C}=\frac{\mathrm{A}}{2p(p+1)(p+2)\phi_1(\mathfrak{a})\phi_3(\mathfrak{a})}, \\
\mathrm{E}=\frac{\mathrm{A}^2}{4p^2(p+1)(p+2)\phi_1(\mathfrak{a})\phi_3(\mathfrak{a})}, & \mathrm{F}=\frac{\mathrm{A}}{2p^2(p+1)\phi_1(\mathfrak{a})\phi_3(\mathfrak{a})}, & \mathrm{G}=\frac{1}{(p+1)^2\phi_2^2(\mathfrak{a})}, \\
\mathrm{H}=\frac{\mathrm{A}^2}{4p^2(p+1)^2\phi_2^2(\mathfrak{a})}, & \mathrm{K}=\frac{\mathrm{A}}{p(p+1)^2\phi_2^2(\mathfrak{a})}. &
\end{array}
$$

Substituting the values of d_2 and d_3 from Lemma 2.10.9 in (6.3.14), we have

$$
\begin{aligned}
& \left|\mathrm{B}d_1d_3+\mathrm{C}d_1d_2^2+\mathrm{E}d_1^3d_2+\mathrm{F}d_1^3-\mathrm{G}d_2^2-\mathrm{H}d_1^4-\mathrm{K}d_1^2d_2\right| \\
= & \Bigg|\frac{1}{4}\mathrm{B}d_1\left\{d_1^3+2d_1(4-d_1^2)x-d_1(4-d_1^2)x^2+2(4-d_1^2)\left(1-|x|^2\right)\mathrm{z}\right\} \\
& +\frac{1}{4}\mathrm{C}d_1\left\{d_1^4+2d_1^2(4-d_1^2)x+(4-d_1^2)^2x^2\right\}+\frac{1}{2}\mathrm{E}d_1^3\left\{d_1^2+(4-d_1^2)x\right\} \\
& +\mathrm{F}d_1^3-\frac{1}{4}\mathrm{G}\left\{d_1^4+2d_1^2(4-d_1^2)x+(4-d_1^2)^2x^2\right\}-\mathrm{H}d_1^4 \\
& -\frac{1}{2}\mathrm{K}d_1^2\left\{d_1^2+(4-d_1^2)x\right\}\Bigg|.
\end{aligned}
$$

Simple computation gives

$$4\left|\mathrm{B}d_1d_3+\mathrm{C}d_1d_2^2+\mathrm{E}d_1^3d_2+\mathrm{F}d_1^3-\mathrm{G}d_2^2-\mathrm{H}d_1^4-\mathrm{K}d_1^2d_2\right| = \Big|(\mathrm{C}+2\mathrm{E})\, d_1^5+$$

$$(\mathrm{B}-\mathrm{G}-\mathrm{H}-2\mathrm{K})\, d_1^4+\mathrm{F}d_1^3+(2\mathrm{B}+2\mathrm{C}d_1+2\mathrm{E}d_1-2\mathrm{G}-2\mathrm{K})\, d_1^2\left(4-d_1^2\right)|x|$$

$$+2\mathrm{B}d_1\left(4-d_1^2\right)\left(1-|x|^2\right)|\mathrm{z}|+\left\{\mathrm{C}d_1\left(4-d_1^2\right)-\mathrm{B}d_1^2-\mathrm{G}\left(4-d_1^2\right)\right\}\left(4-d_1^2\right)|x|^2\Big| \qquad (6.3.15)$$

Applying triangle inequality and replacing $|x|$ by ρ in (6.3.15) , we have

$$4\left|\mathrm{B}d_1d_3+\mathrm{C}d_1d_2^2+\mathrm{E}d_1^3d_2+\mathrm{F}d_1^3-\mathrm{G}d_2^2-\mathrm{H}d_1^4-\mathrm{K}d_1^2d_2\right| \leq (|\mathrm{C}|+2|\mathrm{E}|)\, d_1^5+|\mathrm{F}|\, d_1^3+$$

$$\begin{aligned}
& (\mathrm{B}-\mathrm{G}+|\mathrm{H}|+2|\mathrm{K}|)\, d_1^4+\{2\mathrm{B}+2|\mathrm{C}|\, d_1+2|\mathrm{E}|\, d_1-2\mathrm{G}+2|\mathrm{K}|\}\, d_1^2\left(4-d_1^2\right)\rho \\
& +2\mathrm{B}d_1\left(4-d_1^2\right)\left(1-\rho^2\right)+\left\{|\mathrm{C}|\, d_1\left(4-d_1^2\right)-\mathrm{B}d_1^2-\mathrm{G}\left(4-d_1^2\right)\right\}\left(4-d_1^2\right)\rho^2 \\
= \;& \mathrm{L}(d_1,\rho). \qquad (6.3.16)
\end{aligned}$$

Taking partial derivative of $\mathrm{L}(d_1,\rho)$ with respect to ρ, we have

$$\begin{aligned}
\frac{\partial \mathrm{L}(d_1,\rho)}{\partial\rho} = \;& \{2\mathrm{B}+2|\mathrm{C}|\, d_1+2|\mathrm{E}|\, d_1-2\mathrm{G}+2|\mathrm{F}|\}\, d_1^2\left(4-d_1^2\right)-4\mathrm{B}d_1\left(4-d_1^2\right)\rho \\
& +2\left\{|\mathrm{C}|\, d_1\left(4-d_1^2\right)-\mathrm{B}d_1^2-\mathrm{G}\left(4-d_1^2\right)\right\}\left(4-d_1^2\right)\rho.
\end{aligned}$$

Clearly $\frac{\partial \mathrm{L}(d_1,\rho)}{\partial\rho}>0$, for $0<\rho<1$ and $0<d_1<2$. Therefore, $\mathrm{L}(d_1,\rho)$ is an increasing function of ρ. Also for a fixed $d_1\in[0,2]$, we have

$$\max_{0\leq\rho\leq 1}\mathrm{L}(d_1,\rho)=\mathrm{L}(d_1,\rho)=\mathcal{J}(d_1).$$

Therefore by putting $\rho = 1$ in (6.3.16) , we have

$$\begin{aligned}\mathcal{J}(d_1) &= \{|\mathrm{C}| + 2|\mathrm{E}|\} d_1^5 + \{\mathrm{B} - \mathrm{G} + |\mathrm{H}| + 2|\mathrm{K}|\} d_1^4 + |\mathrm{F}| d_1^3 \\ &\quad + \{2\mathrm{B} + 2|\mathrm{C}| d_1 + 2|\mathrm{E}| d_1 - 2\mathrm{G} + 2|\mathrm{K}|\} d_1^2 \left(4 - d_1^2\right) \\ &\quad + \left\{|\mathrm{C}| d_1 \left(4 - d_1^2\right) - \mathrm{B} d_1^2 - \mathrm{G}\left(4 - d_1^2\right)\right\}\left(4 - d_1^2\right) \end{aligned} \tag{6.3.17}$$

Differentiating (6.3.17) with respect to d_1, we have

$$\begin{aligned}\mathcal{J}'(d_1) &= 5\{|\mathrm{C}| + 2|\mathrm{E}|\} d_1^4 + 4\{\mathrm{B} - \mathrm{G} + |\mathrm{H}| + 2|\mathrm{K}|\} d_1^3 + 3|\mathrm{F}| d_1^2 \\ &\quad + 4\{\mathrm{B} - \mathrm{G} + 2|\mathrm{K}|\} d_1 \left(4 - d_1^2\right) - 4\{\mathrm{B} - \mathrm{G} + 2|\mathrm{K}|\} d_1^3 \\ &\quad + 6\{|\mathrm{C}| + |\mathrm{E}|\} d_1^2 \left(4 - d_1^2\right) - 4\{|\mathrm{C}| + |\mathrm{E}|\} d_1^4 + |\mathrm{C}| d_1^2 \left(4 - d_1^2\right)^2 \\ &\quad - 4|\mathrm{C}| d_1^2 \left(4 - d_1^2\right) - 2\mathrm{B} d_1 \left(4 - d_1^2\right) + 2\mathrm{B} d_1^3 - 4\mathrm{G} d_1 \left(4 - d_1^2\right). \end{aligned} \tag{6.3.18}$$

Again differentiating (6.3.18) with respect to d_1, we have

$$\begin{aligned}\mathcal{J}''(d_1) &= 20\{|\mathrm{C}| + 2|\mathrm{E}|\} d_1^3 + 12\{\mathrm{B} - \mathrm{G} + |\mathrm{H}| + 2|\mathrm{K}|\} d_1^2 \\ &\quad + 6|\mathrm{F}| d_1 + 4\{\mathrm{B} - \mathrm{G} + 2|\mathrm{K}|\}\left(4 - d_1^2\right) - 8\{\mathrm{B} - \mathrm{G} + 2|\mathrm{K}|\} d_1^2 \\ &\quad - 12\{\mathrm{B} - \mathrm{G} + 2|\mathrm{K}|\} d_1^2 + 126\{|\mathrm{C}| + |\mathrm{E}|\}\left(4 - d_1^2\right) - 126\{|\mathrm{C}| + |\mathrm{E}|\} d_1^3 \\ &\quad - 16\{|\mathrm{C}| + |\mathrm{E}|\} d_1^3 + 2|\mathrm{C}| d_1 \left(4 - d_1^2\right)^2 - 4|\mathrm{C}| d_1^3 \left(4 - d_1^2\right)^2 - 8|\mathrm{C}| d_1 \left(4 - d_1^2\right) \\ &\quad + 8|\mathrm{C}| d_1^3 - 2\mathrm{B}\left(4 - d_1^2\right) + 4\mathrm{B} d_1^2 + 6\mathrm{B} d_1^2 - 4\mathrm{G}\left(4 - d_1^2\right) + 8\mathrm{G} d_1^2. \end{aligned}$$

For maximum value of $\mathcal{J}(d_1)$, clearly $\mathcal{J}'(d_1) = 0$ for $d_1 = 0$ and $\mathcal{J}''(0) < 0$, so $\mathcal{J}(d_1)$ has maximum value at $d_1 = 0$ hence

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \left[\frac{4|\mathfrak{b}|\,\mathfrak{c}(\mathfrak{c}+1)(1-\lambda)}{(p+1)(\mathfrak{a}+1)}\right]^2.$$

This completes the proof. $\square$

Inclusion Results of the Function Class $\mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$

Theorem 6.3.5. Let $\lambda > p$, $\mathfrak{b} \in \mathbb{C}\diagdown\{0\}$. Then

$$\mathcal{SC}_p^0(\mathfrak{a}+1,\mathfrak{b},\mathfrak{c},\lambda) \subset \mathcal{SC}_p^0(\mathfrak{a},\mathfrak{b}+1,\mathfrak{c},\lambda_1),$$

where

$$\lambda_1 = \frac{\mathfrak{b}(\mathfrak{a}+1)}{\mathfrak{a}(\mathfrak{b}+1)}\lambda - \frac{\mathfrak{b}-\mathfrak{a}}{\mathfrak{a}(\mathfrak{b}+1)}.$$

Proof. Suppose $f \in \mathcal{SC}_p^0(\mathfrak{a}+1,\mathfrak{b},\mathfrak{c},\lambda)$ and set

$$1-\frac{2}{\mathfrak{b}+1}+\frac{2}{\mathfrak{b}+1}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})} = h(\mathrm{z}), \tag{6.3.19}$$

Logarithmic differentiation of (6.3.19) , gives

$$\frac{\mathrm{z}\left(\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})\right)'}{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})} - \frac{\mathrm{z}\left(\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})\right)'}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})} = \frac{(\mathfrak{b}+1)\,\mathrm{z}h'(\mathrm{z})}{(\mathfrak{b}+1)\left\{h(\mathrm{z})-1\right\}+2}.$$

Using the identity(2.9.3) we have

$$1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+2,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})} = 1-\frac{\mathfrak{a}}{\mathfrak{a}+1}\frac{\mathfrak{b}+1}{\mathfrak{b}}+\frac{\mathfrak{a}}{\mathfrak{a}+1}\frac{\mathfrak{b}+1}{\mathfrak{b}}h(\mathrm{z})+\frac{2}{\mathfrak{b}(\mathfrak{a}+1)}\frac{\mathrm{z}h'(\mathrm{z})}{h(\mathrm{z})-1+\frac{2}{\mathfrak{b}+1}}. \tag{6.3.20}$$

Let

$$1-\frac{\mathfrak{a}}{\mathfrak{a}+1}\frac{\mathfrak{b}+1}{\mathfrak{b}}+\frac{\mathfrak{a}}{\mathfrak{a}+1}\frac{\mathfrak{b}+1}{\mathfrak{b}}h(\mathrm{z}) = H(\mathrm{z}).$$

From (6.3.20) , we have

$$1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+2,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})} = H(\mathrm{z})+\frac{\mathrm{z}H'(\mathrm{z})}{\mu_1 H(\mathrm{z})+\mu_2},$$

where $\mu_3 = \frac{\mathfrak{b}(\mathfrak{a}+1)}{2}$ and $\mu_4 = \frac{2\mathfrak{a}-\mathfrak{a}\mathfrak{b}-\mathfrak{b}}{2}$. Since $f \in \mathcal{SC}_p^0(\mathfrak{a}+1,\mathfrak{b},\mathfrak{c},\lambda)$, from Lemma 2.10.10 we have

$$H(\mathrm{z})+\frac{\mathrm{z}H'(\mathrm{z})}{\mu_3 H(\mathrm{z})+\mu_4} = 1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+2,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})} \prec q(\mathrm{z}),$$

where $q(\mathrm{z})$ is given by

$$q(\mathrm{z}) = \frac{1-(2\lambda-1)\,\mathrm{z}}{1-\mathrm{z}}. \tag{6.3.21}$$

Applying Lemma 6.2.1 we have

$$H(\mathrm{z}) \prec q(\mathrm{z})$$

or equivalently

$$h(\mathrm{z}) \prec \frac{1-(2\lambda_1-1)\,\mathrm{z}}{1-\mathrm{z}},$$

where

$$\lambda_1 = \frac{\mathfrak{b}(\mathfrak{a}+1)}{\mathfrak{a}(\mathfrak{b}+1)}\lambda - \frac{\mathfrak{b}-\mathfrak{a}}{\mathfrak{a}(\mathfrak{b}+1)}.$$

This completes the proof. □

Theorem 6.3.6. Let $f \in \mathcal{SC}_p^0(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \lambda)$. Then $\mathfrak{I} \in \mathcal{SC}_p^0(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \lambda)$, where $\mathfrak{I}$ is defined by

$$\mathfrak{I}(\mathrm{z}) = \frac{\eta+p}{\mathrm{z}^\eta}\int_0^{\mathrm{z}} t^{\eta-1}f(t)dt, \quad \eta > -p. \tag{6.3.22}$$

Proof. Suppose

$$1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+2,\mathfrak{c})\mathfrak{I}(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})\mathfrak{I}(\mathrm{z})} = h(\mathrm{z}),$$

Now differentiating (6.3.22) , we have

$$(\eta+p)f(\mathrm{z}) = \eta\mathfrak{I}(\mathrm{z}) + \mathrm{z}\mathfrak{I}'(\mathrm{z}).$$

Applying the operator $\mathbb{k}_p(\mathfrak{a}, \mathfrak{c})$ we have

$$(\eta+p)\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z}) = \eta\mathbb{k}_p(\mathfrak{a},\mathfrak{c})\mathfrak{I}(\mathrm{z}) + \mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})\mathfrak{I}(\mathrm{z}), \tag{6.3.23}$$

and

$$(\eta+p)\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z}) = \eta\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})\mathfrak{I}(\mathrm{z}) + \mathbb{k}_p(\mathfrak{a}+2,\mathfrak{c})\mathfrak{I}(\mathrm{z}) \tag{6.3.24}$$

From (6.3.23) and (6.3.24) , we have

$$\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}=\frac{\eta\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}+\frac{\mathbb{k}_p(\mathfrak{a}+2,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}}{\eta+\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}}. \tag{6.3.25}$$

Logarithmic differentiation of (6.3.23) , together with (2.9.3) and (6.3.25) , we have

$$1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}=h(\mathrm{z})+\frac{\mathrm{z}h'(\mathrm{z})}{\mu_3 h(\mathrm{z})+\mu_4},$$

where $\mu_3=\frac{\mathfrak{a}\mathfrak{b}}{2}$ and $\mu_4=\eta+p-\frac{\mathfrak{a}\mathfrak{b}}{2}$.

Since $f\in\mathcal{SC}_p^0(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$, from Lemma 6.2.1, we have

$$h(\mathrm{z})+\frac{\mathrm{z}h'(\mathrm{z})}{\mu_3 h(\mathrm{z})+\mu_4}=1-\frac{2}{\mathfrak{b}}+\frac{2}{\mathfrak{b}}\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}\prec q(\mathrm{z}),$$

where $q(\mathrm{z})$ is given by (6.3.21).

Applying Lemma 2.10.10 we have

$$h\prec q.$$

Which implies that $\Im\in\mathcal{SC}_p^0(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. This completes the proof. □

6.3 Conclusion

In this chapter, we have introduced a new subclass $\mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$ of multivalent functions of complex order by using the operator $\mathbb{k}_p$, and have studied some interesting properties such as coefficient estimates, sufficiency criteria, Fekete-Szego inequality, inclusion result and integral preserving property for this class. The connection between newly defined classes and a number of already known classes of univalent and multivalent functions are obtained. Assigning particular values to different parameters in our main results, several known results are deduced.

Chapter 7

Some Convolution Properties of a Subclass of Analytic Functions

7.1 Introduction

Let $\mathcal{A}(p)$ be the class of all multivalent functions of the form (2.4.1) and as discussed in section 2.4. In 1977, Chichra [9] introduced and studied the class $\mathcal{R}$, the family of functions $f \in \mathcal{A}$ satisfy the condition

$$\mathfrak{Re}\,(f'(\mathrm{z})+\mathrm{z}f''(z)) > 0, \qquad \mathrm{z} \in \mathbb{E}.$$

Chichra proved that if $f \in \mathcal{R}$, then $\mathfrak{Re} f'(\mathrm{z}) > 0$ in $\mathbb{E}$, and hence f is univalent in $\mathbb{E}$. If $f \in \mathcal{R}$, then f is also starlike in $\mathbb{E}$, see [117].

In this chapter we define a new class of multivalent functions by using the idea of Chichra, and improve the results given in [9]. We shall also get a new criterion for convolution properties of functions in this class.

Definition 7.1.1. Let $f \in \mathcal{A}(p)$. Then $f \in \mathcal{R}_p(\beta)$, if

$$\mathfrak{Re}\left\{e^{i\beta}\left(\frac{f^{(p)}(\mathrm{z})+\mathrm{z}f^{(p+1)}(\mathrm{z})}{p!}\right)\right\} \prec \cos\beta\,(h(\mathrm{z}))+i\sin\beta, \quad (\mathrm{z} \in \mathbb{E},\ p \in \mathbb{N} = \{1,2,...\}).$$

where $f^{(p)}$ is the pth derivative of f and $h(\mathrm{z}) = \frac{1+\mathrm{z}}{1-\mathrm{z}}$ with $h(0) = 1$.

For $\beta = 0$ we have the following definition which is the main motivation of this chapter.

Definition 7.1.2. A function $f \in \mathcal{A}(p)$ given in (2.4.1) is said to belongs to the class $\mathcal{R}_p$, if

$$\mathfrak{Re}\left(\frac{f^{(p)}(\mathrm{z})+\mathrm{z}f^{(p+1)}(\mathrm{z})}{p!}\right) > 0, \qquad (\mathrm{z} \in \mathbb{E},\ p \in \mathbb{N} = \{1,2,...\}),$$

where $f^{(p)}$ is the pth derivative of f. When $p = 1$, we obtain $\mathcal{R}_1 = \mathcal{R}$, studied in [9, 117].

7.2 Main Results

Theorem 7.2.1. Let $f \in \mathcal{R}_p$, then

$$\mathfrak{Re}\left(\frac{f^{(p)}(\mathrm{z})}{p!}\right) > -1+2\log 2, \qquad (\mathrm{z} \in \mathbb{E}).$$

The constant $-1+2\log 2$ cannot be improved.

Proof. Let $f \in \mathcal{R}_p$, then we have

$$\mathfrak{Re}\left(1+\sum_{\ell=1}^{\infty}\frac{(p+\ell)!(\ell+1)}{p!\ell!}a_{p+\ell}\mathrm{z}^{\ell}\right) > 0, \qquad (\mathrm{z} \in \mathbb{E}), \tag{7.2.1}$$

or, equivalently

$$\mathfrak{Re}\left(1+\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(p+\ell)!(\ell+1)}{p!\ell!}a_{p+\ell}\mathrm{z}^{\ell}\right) > \frac{1}{2}, \qquad (\mathrm{z} \in \mathbb{E}). \tag{7.2.2}$$

Consider the function

$$h(\mathrm{z}) = 1+2\sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\mathrm{z}^{\ell}. \tag{7.2.3}$$

From (7.2.3) we have

$$\begin{aligned}\mathfrak{Re}h(\mathrm{z}) &= \mathfrak{Re}\left(1-\frac{2}{\mathrm{z}}\{\mathrm{z}+\log(1-\mathrm{z})\}\right)\\ &> -1+2\log 2, \quad (\text{see [103]})\,. \end{aligned}\tag{7.2.4}$$

From (7.2.2) and (7.2.3) we obtain

$$\frac{f^{(p)}(\mathrm{z})}{p!} = \left(1+\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(p+\ell)!(\ell+1)}{p!\ell!}a_{p+\ell}\mathrm{z}^{\ell}\right) * \left(1+2\sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\mathrm{z}^{\ell}\right).$$

From which it follows, in view of (7.2.2), (7.2.4) and Lemma 2.10.13 that

$$\mathfrak{Re}\left(\frac{f^{(p)}(\mathrm{z})}{p!}\right) > -1+2\log 2, \qquad (\mathrm{z} \in \mathbb{E}).$$

The constant $-1+2\log 2$ cannot be replaced by any larger one follows from the fact that the function f_1 defined by $\frac{\mathrm{z}f_1^{(p)}(\mathrm{z})}{p!} = -\mathrm{z}-2\log(1-\mathrm{z})$ is in the class $\mathcal{R}_p$. $\square$

Corollary 7.2.1. [117] If $f \in \mathcal{R}$, then

$$\mathfrak{Re}f'(\mathrm{z}) > -1+2\log 2 = 0.39..., \qquad (\mathrm{z} \in \mathbb{E}).$$

The constant $-1+2\log 2$ cannot be replaced by any larger one.

Theorem 7.2.2. Let $f \in \mathcal{R}_p$, then

$$\mathfrak{Re}\left(\frac{f^{(p-1)}(\mathrm{z})}{\mathrm{z}}\right) > \frac{p!}{2}\ , \qquad (\mathrm{z} \in \mathbb{E}).$$

Proof. Sine the sequence $\{d_\ell\}_0^\infty$ defined by $d_0 = 1$, $d_\ell = \frac{2}{(\ell+1)^2}$, $\ell \geq 1$ is a convex null sequence, using Lemma 2.10.11, we have

$$\mathfrak{Re}\left(1+2\sum_{\ell=1}^{\infty}\frac{1}{(\ell+1)^2}\mathrm{z}^\ell\right) > \frac{1}{2}\ , \qquad (\mathrm{z} \in \mathbb{E}). \tag{7.2.5}$$

We can write

$$\frac{f^{(p-1)}(\mathrm{z})}{p!\mathrm{z}} = \left(1+\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(p+\ell)!(\ell+1)}{p!\ell!}a_{p+\ell}\mathrm{z}^\ell\right) * \left(1+2\sum_{\ell=1}^{\infty}\frac{1}{(\ell+1)^2}\mathrm{z}^\ell\right).$$

The result follows at once from (7.2.2), (7.2.5) and Lemma 2.10.13. □

Theorem 7.2.3. Let $f \in \mathcal{R}_p$, then, for every $\ell \geq 1$, the nth partial sum of f, satisfies $\mathfrak{Re} S_\ell^{(p)}(\mathrm{z}, f) > 0$, $\mathrm{z} \in \mathbb{E}$, and hence $S_\ell(\mathrm{z}, f)$ is p-valent in $\mathbb{E}$.

Proof. From (7.2.1) and (7.2.3) we can write

$$\frac{S_\ell^{(p)}(\mathrm{z}, f)}{p!} = \left(1+\sum_{\ell=1}^{\infty}\frac{(p+\ell)!(\ell+1)}{p!\ell!}a_{p+\ell}\mathrm{z}^\ell\right) * \left(1+\sum_{\ell=1}^{\infty}\frac{1}{(\ell+1)}\mathrm{z}^\ell\right), \tag{7.2.6}$$

putting $\mathrm{z} = \mathrm{r}e^{i\theta}$, $0 \leq \mathrm{r} \leq 1$, $0 \leq |\theta| \leq \pi$, and using the minimum principle for harmonic functions with the result see [106] we have

$$
\begin{aligned}
\mathfrak{Re}\left(1+\sum_{\ell=1}^{k}\frac{z^{\ell}}{\ell+1}\right) &= \mathfrak{Re}\left(1+\sum_{\ell=1}^{k}\frac{r^{\ell}e^{i\ell\theta}}{\ell+1}\right) \\
&= \mathfrak{Re}\left(1+\sum_{\ell=1}^{k}\frac{r^{\ell}}{\ell+1}(\cos\ell\theta+i\sin\ell\theta)\right) \\
&= 1+\sum_{\ell=1}^{k}\frac{r^{\ell}}{\ell+1}\cos\ell\theta \quad (0\leq\theta\leq\pi) \\
&= 1+\sum_{\ell=1}^{k}\frac{\cos\ell\theta}{\ell+1}\geq\frac{1}{2}. \qquad (7.2.7)
\end{aligned}
$$

Using (7.2.1), (7.2.6), (7.2.7) and Lemma 2.10.13, we have $\mathfrak{Re}\left(S_{\ell}^{(p)}(z,f)\right)>0,\ \ z\in\mathbb{E}$. From the result given in [91], we see that $S_{\ell}(z,f)$ is p-valent in $\mathbb{E}$ for every $\ell\geq 1$. $\square$

If $p=1$, then Theorem 7.2.2 and Theorem 7.2.3 were proved in [117].

Theorem 7.2.4. If $f\in\mathcal{A}(p)$ and

$$
\mathfrak{Re}\left(\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}\right)>-\frac{1}{4}, \qquad (z\in\mathbb{E}), \tag{7.2.8}
$$

then $f\in\mathcal{S}_p^{*}(p-1)$.

Proof. Let $f\in\mathcal{A}(p)$ given by (2.4.1) , it follows from the hypothesis of the theorem that

$$
\mathfrak{Re}\left(1+\frac{2}{5}\sum_{\ell=1}^{\infty}\frac{(p+\ell)!(\ell+1)}{p!\ell!}a_{p+\ell}z^{\ell}\right)>\frac{1}{2}, \qquad (z\in\mathbb{E}). \tag{7.2.9}
$$

Also the sequence $\{d_{\ell}\}_0^{\infty}$, where $d_0=1$ and $d_{\ell}=\frac{5}{2}\frac{1}{(\ell+1)^2}$, $\ell\geq 1$ is a convex null sequence such as

$$
\mathfrak{Re}\left(1+\frac{5}{2}\sum_{\ell=1}^{\infty}\frac{1}{(\ell+1)^2}z^{\ell}\right)>\frac{1}{2}, \qquad (z\in\mathbb{E}). \tag{7.2.10}
$$

From (7.2.9), (7.2.10) and Lemma 2.10.11, we obtain

$$\begin{aligned}\mathfrak{Re}\left(\frac{f^{(p-1)}(\mathrm{z})}{p!\mathrm{z}}\right) &= \mathfrak{Re}\left(1+\frac{2}{5}\sum_{\ell=1}^{\infty}\frac{(p+\ell)!(\ell+1)}{p!\ell!}a_{p+\ell}\mathrm{z}^{\ell}\right) * \left(1+\frac{5}{2}\sum_{\ell=1}^{\infty}\frac{1}{(\ell+1)^{2}}\mathrm{z}^{\ell}\right)\\ &> \frac{1}{2}, \qquad (\mathrm{z}\in\mathbb{E}). \qquad (7.2.11)\end{aligned}$$

Now we define a function w by

$$\frac{\mathrm{z}f^{(p)}(\mathrm{z})}{f^{(p-1)}(\mathrm{z})} = \frac{1+w(\mathrm{z})}{1-w(\mathrm{z})} \qquad (\mathrm{z}\in\mathbb{E}). \qquad (7.2.12)$$

Clearly $w(0)=0$. Since f is p-valent in $\mathbb{E}$, we have $w(\mathrm{z})\neq 1$ in $\mathbb{E}$. From (7.2.12) we obtain

$$\frac{f^{(p)}(\mathrm{z})+\mathrm{z}f^{(p+1)}(\mathrm{z})}{p!} = \frac{f^{(p-1)}(\mathrm{z})}{p!\mathrm{z}}\left(\left(\frac{1+w(\mathrm{z})}{1-w(\mathrm{z})}\right)^{2}+\frac{2\mathrm{z}w'(\mathrm{z})}{(1-w(\mathrm{z}))^{2}}\right). \qquad (7.2.13)$$

We claim that $|w(\mathrm{z})|<1$ in $\mathbb{E}$. If this is not true, then there exists a point $\mathrm{z}_\circ\in\mathbb{E}$ such that

$$max_{|\mathrm{z}|\leq|\mathrm{z}_\circ|}|w(\mathrm{z})| = |w(\mathrm{z}_\circ)| = 1.$$

Then, from the result given in [28], we have

$$\mathrm{z}w'(\mathrm{z}_\circ) = kw(\mathrm{z}_\circ), \text{ where } k\geq 1,\ w(\mathrm{z}_\circ)=e^{i\theta},\ 0<\theta<2\pi.$$

Putting $\mathrm{z}=\mathrm{z}_\circ$ in (7.2.13), we obtain

$$\begin{aligned}\mathfrak{Re}\left(\frac{f^{(p)}(\mathrm{z}_\circ)+\mathrm{z}_\circ f^{(p+1)}(\mathrm{z}_\circ)}{p!}\right) &= \mathfrak{Re}\left(\frac{f^{(p-1)}(\mathrm{z}_\circ)}{p!\mathrm{z}_\circ}\left(\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^{2}+\frac{2ke^{i\theta}}{(1-e^{i\theta})^{2}}\right)\right)\\ &\leq -\frac{\cos\theta+1+k}{\cos\theta-1}\mathfrak{Re}\left(\frac{f^{(p-1)}(\mathrm{z}_\circ)}{(\mathrm{z}_\circ)}\right)\end{aligned}$$

This can be written as

$$\begin{aligned}\mathfrak{Re}\left(\frac{f^{(p)}(\mathrm{z}_\circ)+\mathrm{z}_\circ f^{(p+1)}(\mathrm{z}_\circ)}{p!}\right) &\leq -\frac{k}{2}\mathfrak{Re}\left(\frac{f^{(p-1)}(\mathrm{z}_\circ)}{(\mathrm{z}_\circ)}\right)\\ &\leq -\frac{1}{4}. \end{aligned} \tag{7.2.14}$$

Since $k \geq 1$ and from (7.2.11), $\mathfrak{Re}\left(\frac{f^{(p-1)}(\mathrm{z})}{p!\mathrm{z}}\right) > \frac{1}{2}$, $\mathrm{z} \in \mathbb{E}$. Inequality (7.2.14) contradicts inequality (7.2.8), thus $|w(\mathrm{z})| < 1$ in $\mathbb{E}$. Equation (7.2.12) then implies that $f \in \mathcal{S}_p^*(p-1)$ (see [64]) . $\square$

Corollary 7.2.2. [117]**.** If $f \in \mathcal{A}$, and let

$$\mathfrak{Re}\left(\frac{f^{'}(\mathrm{z}) + \mathrm{z}f^{''}(\mathrm{z})}{p!}\right) > -\frac{1}{4}, \qquad (\mathrm{z} \in \mathbb{E}),$$

then $f \in \mathcal{S}^*$.

Using the Alexander type relation, we obtain the following corollary.

Corollary 7.2.3. If $g \in \mathcal{A}(p)$, and

$$\mathfrak{Re}\left(\frac{g^{(p)}(\mathrm{z}) + 3\mathrm{z}g^{(p+1)}(\mathrm{z}) + \mathrm{z}^2g^{(p+2)}(\mathrm{z})}{p!}\right) > -\frac{1}{4}, \qquad (\mathrm{z} \in \mathbb{E}),$$

then $g \in \mathcal{C}_p(p-1)$.

Our next result shows that the class $\mathcal{R}_p$ is closed with respect to Hadamard product.

Theorem 7.2.5. If f and g belong to the class $\mathcal{R}_p$ and

$$H^{(p-1)}(\mathrm{z}) = f^{(p-1)}(\mathrm{z}) * g^{(p-1)}(\mathrm{z}),$$

then H is also belongs to the class $\mathcal{R}_p$.

Proof. Since

$$H^{(p-1)}(\mathrm{z}) = f^{(p-1)}(\mathrm{z}) * g^{(p-1)}(\mathrm{z}),$$

from above we obtain

$$\mathrm{z}H^{(p)}(\mathrm{z}) = \mathrm{z}f^{(p)}(\mathrm{z}) * g^{(p-1)}(\mathrm{z}).$$

A simple computation gives

$$\mathfrak{Re}\left(\frac{H^{(p)}(\mathrm{z})+\mathrm{z}H^{(p+1)}(\mathrm{z})}{p!}\right) = \mathfrak{Re}\left(\left(\frac{f^{(p)}(\mathrm{z})+\mathrm{z}f^{(p+1)}(\mathrm{z})}{p!}\right) * \left(\frac{g^{(p-1)}(\mathrm{z})}{p!\mathrm{z}}\right)\right). \qquad (7.2.15)$$

From (7.2.15) , using (2.4.1) , (7.2.11) and Lemma 2.10.13, we have the desired result. □

Corollary 7.2.4. [117]**.** If $f(z) = \mathrm{z}+\sum_{\ell=2}^{\infty} a_\ell \mathrm{z}^\ell$ and $g(\mathrm{z}) = \mathrm{z}+\sum_{\ell=2}^{\infty} b_\ell \mathrm{z}^\ell$ belong to $\mathcal{R}$ then their Hadamard product

$$H(\mathrm{z}) = (f * g)(\mathrm{z})$$

also belongs to $\mathcal{R}$.

Theorem 7.2.6. If $f \in \mathcal{R}_p$, $g \in \mathcal{A}(p)$ and $Re\left(\frac{g^{(p-1)}(\mathrm{z})}{p!\mathrm{z}}\right) > \frac{1}{2}$, $\mathrm{z} \in \mathbb{E}$, then

$$f^{(p-1)}(\mathrm{z}) * g^{(p-1)}(\mathrm{z})$$

also belongs to the class $\mathcal{R}_p$.

Theorem 7.2.7. Let $f \in \mathcal{R}_p$. Then $Re\left(f^{(p)}(\mathrm{z})\right) > 0$, $\mathrm{z} \in \mathbb{E}$, and hence f is p-valent in $\mathbb{E}$.

Proof. Using the result given in [91] and application of Lemma 2.10.12 with $\vartheta(\mathrm{z}) = \frac{\mathrm{z}f^{(p)}(\mathrm{z})}{p!}$ and $\chi(\mathrm{z}) = \mathrm{z}$ proves Theorem 7.2.7. □

If $p = 1$, then Theorem 3.7 was proved earlier in [9].

In [110], it is shown that the class $\mathcal{C}$ is closed with respect to Hadamard product. In the next result we will prove that the Hadamard product of functions of the class $\mathcal{R}_p$ belong to the class $\mathcal{C}_p(\lambda)$.

Theorem 7.2.8. If f and g belong to $\mathcal{R}_p$, and

$$H^{(p-1)}(\mathrm{z}) = f^{(p-1)}(\mathrm{z}) * g^{(p-1)}(\mathrm{z}),$$

then $H \in \mathcal{C}_p(p-1)$.

Proof. In view of Corollary 7.2.3 it is sufficient to show that

$$\mathfrak{Re}\left(\frac{H^{(p)}(\mathrm{z})+3\mathrm{z}H^{(p+1)}(\mathrm{z})+\mathrm{z}^2H^{(p+2)}(\mathrm{z})}{p!}\right) > -\frac{1}{4}, \qquad (\mathrm{z} \in \mathbb{E}).$$

Equivalently this can be written as

$$\mathfrak{Re}\left(1+\sum_{\ell=1}^{\infty}(\ell+1)\left(\frac{(\ell+p)!}{\ell!p!}\right)^2 a_{\ell+p}b_{\ell+p}\mathrm{z}^{\ell}\right) > -\frac{1}{4}, \qquad (\mathrm{z} \in \mathbb{E}). \qquad (7.2.16)$$

Since f, $g \in \mathcal{R}_p$, we have

$$\mathfrak{Re}\left(1+\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(\ell+p)!(\ell+1)}{p!\ell!}a_{\ell+p}\mathrm{z}^{\ell}\right) > \frac{1}{2}, \qquad (\mathrm{z} \in \mathbb{E}),$$

and

$$\left(1+\frac{1}{2}\sum_{\ell=1}^{\infty}\frac{(\ell+p)!(\ell+1)}{p!\ell!}b_{\ell+p}\mathrm{z}^{\ell}\right) > \frac{1}{2}, \qquad (\mathrm{z} \in \mathbb{E}).$$

Using Lemma 2.10.13, we obtain

$$\mathfrak{Re}\left(1+\frac{1}{4}\sum_{\ell=1}^{\infty}(\ell+1)^2\left(\frac{(\ell+p)!}{\ell!p!}\right)^2 a_{\ell+p}b_{\ell+p}\mathrm{z}^{\ell}\right) > \frac{1}{2}, \qquad (\mathrm{z} \in \mathbb{E}). \qquad (7.2.17)$$

Consider the function

$$h(\mathrm{z}) = \left(1+4\sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\mathrm{z}^{\ell}\right). \qquad (7.2.18)$$

From (7.2.18), we have

$$\mathfrak{Re}\left(1+4\sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\mathrm{z}^{\ell}\right) = \mathfrak{Re}\left(-3-\frac{4}{\mathrm{z}}\log(1-\mathrm{z})\right)$$
$$> -3+4\log 2 \quad \text{(see [103])}$$
$$> -\frac{1}{4}, \qquad (\mathrm{z}\in\mathbb{E})\,. \tag{7.2.19}$$

From (7.2.17) and (7.2.18), we can write

$$\begin{aligned}&\left(1+\sum_{\ell=1}^{\infty}(\ell+1)\left(\frac{(\ell+p)!}{\ell!p!}\right)^{2}a_{\ell+p}b_{\ell+p}\mathrm{z}^{\ell}\right)\\ =\ &\left(1+\frac{1}{4}\sum_{\ell=1}^{\infty}(\ell+1)^{2}\left(\frac{(\ell+p)!}{\ell!p!}\right)^{2}a_{\ell+p}b_{\ell+p}\mathrm{z}^{\ell}\right)*\left(1+4\sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\mathrm{z}^{\ell}\right).\end{aligned} \tag{7.2.20}$$

Using $(7.2.17)\,,(7.2.19)\,,(7.2.20)$ and Lemma 2.10.13 we see that (7.2.16) holds for all $\mathrm{z}\in\mathbb{E}$. This completes the proof. $\square$

Corollary 7.2.5. [117] If $f(z)=\mathrm{z}+\sum_{\ell=2}^{\infty}a_{\ell}\mathrm{z}^{\ell}$ and $g(\mathrm{z})=\mathrm{z}+\sum_{\ell=2}^{\infty}b_{\ell}\mathrm{z}^{\ell}$ belong to $\mathcal{R}$, then

$$H(\mathrm{z})=(f*g)(\mathrm{z})\in\mathcal{C}.$$

In the next result we prove that the class $\mathcal{R}_p$ preserved under the integral operator, defined as:

Let $f\in\mathcal{A}(p)$ and be given by $(2.4.1)$. We define an integral operator by

$$\begin{aligned}\frac{\mathfrak{I}^{(p-1)}(\mathrm{z})}{p!} &= \frac{\eta+1}{\mathrm{z}^{\eta}}\int_{0}^{\mathrm{z}}t^{\eta-1}\left(\frac{f^{(p-1)}(t)}{p!}\right)dt, \qquad (\eta>-1,\ p\in N=\{1,2,...\}).\\ &= \mathrm{z}+\sum_{\ell=1}^{\infty}\frac{(\eta+1)\,(\ell+p)!}{(\ell+\eta+1)\,(\ell+1)!p!}a_{\ell+p}\mathrm{z}^{\ell+1}.\end{aligned} \tag{7.2.21}$$

For $p=1$ the operator defined in (7.2.21) is a generalized form of operator by Bernard [4]. For comprehensive study of operators with applications, we refer [7, 12, 18, 50, 68, 76].

Theorem 7.2.9. Let $f \in \mathcal{R}_p$ and

$$\frac{\mathfrak{I}^{(p-1)}(\mathrm{z})}{p!} = \frac{\eta+1}{\mathrm{z}^{\eta}} \int_0^{\mathrm{z}} t^{\eta-1} \left(\frac{f^{(p-1)}(t)}{p!} \right) dt \qquad (\mathrm{z} \in \mathbb{E}). \tag{7.2.22}$$

Then $\mathfrak{I} \in \mathcal{R}_p$.

Proof. Let

$$\frac{\mathfrak{I}^{(p)}(\mathrm{z})+\mathrm{z}\mathfrak{I}^{(p+1)}(\mathrm{z})}{p!} = h(\mathrm{z}).$$

From (7.2.22) , we have

$$\left(\frac{\mathrm{z}^{\eta}\mathfrak{I}^{(p-1)}(\mathrm{z})}{p!} \right)' = (\eta+1)\mathrm{z}^{\eta-1} \frac{f^{(p-1)}(\mathrm{z})}{p!}.$$

A simple computation gives us

$$\mathfrak{Re} \left(h(\mathrm{z})+\frac{\mathrm{z}h'(\mathrm{z})}{\eta+1} \right) = \mathfrak{Re} \left(\frac{f^{(p)}(\mathrm{z})+\mathrm{z}f^{(p+1)}(\mathrm{z})}{p!} \right).$$

From the hypothesis of Theorem 7.2.9 with the result given in [55], we have

$$\mathfrak{Re} \left(\frac{\mathfrak{I}^{(p)}(\mathrm{z})+\mathrm{z}\mathfrak{I}^{(p+1)}(\mathrm{z})}{p!} \right) > 0, \qquad (\mathrm{z} \in \mathbb{E}).$$

This complete the proof. $\square$

It is easy to see that, if $0 \leq \lambda \leq 1$ and f and g are in $\mathcal{R}_p$, then $G(\mathrm{z}) = \lambda g(\mathrm{z})+(1-\lambda)f(\mathrm{z})$ is also in $\mathcal{R}_p$. This shows that the class $\mathcal{R}_p$ is a convex set.

Theorem 7.2.10. Let $f,\, g \in \mathcal{A}(p)$ and $\alpha,\, \beta < 1$.If

$$\frac{f^{(p)}(\mathrm{z})}{p!} \in P(\alpha), \quad \frac{g^{(p)}(\mathrm{z})}{p!} \in P(\beta),$$

and

$$\phi^{(p-1)}(\mathrm{z}) = f^{(p-1)}(\mathrm{z}) * g^{(p-1)}(\mathrm{z}),$$

then $\phi \in \mathcal{S}_p^*(p-1)$, provided that

$$(1-\alpha)(1-\beta) < \frac{3}{8(\ln 2-1)^2+4}. \tag{7.2.23}$$

Proof. Using given hypothesis on f and g and Lemma 2.10.15 we have

$$\begin{aligned} \mathfrak{Re}\left(\frac{f^{(p)}(\mathrm{z})}{p!} * \frac{g^{(p)}(\mathrm{z})}{p!}\right) &= \mathfrak{Re}\left(\frac{\phi^{(p)}(\mathrm{z})+\mathrm{z}\phi^{(p+1)}(\mathrm{z})}{p!}\right) \\ &> 1-2(1-\alpha)(1-\beta). \end{aligned} \tag{7.2.24}$$

By using Lemma 2.10.14, from (7.2.24) we have

$$\mathfrak{Re}\left(\frac{\phi^{(p)}(\mathrm{z})}{p!}\right) > 1+4(1-\alpha)(1-\beta)(\ln 2-1) \qquad (\mathrm{z} \in \mathbb{E}),$$

or, equivalently this can be written as

$$\mathfrak{Re}\left(\frac{\phi^{(p-1)}(\mathrm{z})}{p!\mathrm{z}}+\mathrm{z}\left(\frac{\phi^{(p-1)}(\mathrm{z})}{p!\mathrm{z}}\right)'\right) > 1+4(1-\alpha)(1-\beta)(\ln 2-1) \quad (\mathrm{z} \in \mathbb{E}). \tag{7.2.25}$$

Using Lemma 2.10.14, (7.2.25) becomes

$$\mathfrak{Re}\left(\frac{\phi^{(p-1)}(\mathrm{z})}{p!\mathrm{z}}\right) > 1-8(1-\alpha)(1-\beta)(\ln 2-1)^2. \tag{7.2.26}$$

Suppose

$$h(\mathrm{z}) = \frac{\mathrm{z}\phi^{(p)}(\mathrm{z})}{\phi^{(p-1)}(\mathrm{z})} \text{ and } q(\mathrm{z}) = \frac{\phi^{(p-1)}(\mathrm{z})}{\mathrm{z}}.$$

Then

$$\mathfrak{Re}q(\mathrm{z}) > 1-8(1-\alpha)(1-\beta)(\ln 2-1)^2. \tag{7.2.27}$$

A simple computation gives us

$$\begin{aligned}\phi^{(p)}(\mathrm{z})+\mathrm{z}\phi^{(p+1)}(\mathrm{z}) &= q(\mathrm{z})\left(h^2(\mathrm{z})+\mathrm{z}h'(\mathrm{z})\right)\\ &= \Psi\left(h(\mathrm{z}),\mathrm{z}h'(\mathrm{z}),\mathrm{z}\right). \qquad (7.2.28)\end{aligned}$$

By taking $u = h(\mathrm{z})$ and $v = \mathrm{z}h'(\mathrm{z})$, $\Psi(u, v; \mathrm{z}) = q(\mathrm{z})(u^2+v)$. From (7.2.24) and (7.2.28) we have

$$\mathfrak{Re}\left(\Psi\left(h(\mathrm{z}),\mathrm{z}h'(\mathrm{z}),\mathrm{z}\right)\right) > 1-2(1-\alpha)(1-\beta) \qquad (\mathrm{z}\in\mathbb{E}).$$

Now for real $x, y \le -\frac{1}{2}(1+x^2)$, we have

$$\begin{aligned}\mathfrak{Re}\left(\Psi(ix, y, \mathrm{z})\right) &= \left(-x^2+y\right)\mathfrak{Re}q(\mathrm{z})\\ &\le -\frac{1}{2}\left(1+3x^2\right)\mathfrak{Re}q(\mathrm{z})\\ &\le -\frac{1}{2}\mathfrak{Re}q(\mathrm{z}) \qquad (\mathrm{z}\in\mathbb{E}). \qquad (7.2.29)\end{aligned}$$

From (7.2.27) and (7.2.29) we obtain

$$\mathfrak{Re}\left(\Psi(ix, y, \mathrm{z})\right) \le 1-2(1-\alpha)(1-\beta), \quad \text{for all } \mathrm{z}\in\mathbb{E}.$$

Thus by Lemma 2.10.16, $\mathrm{Re}h(z) > 0$, in $\mathbb{E}$, that is $\phi(\mathrm{z}) \in \mathcal{S}_p^*(p-1)$ see, [64]. □

Corollary 7.2.6. Let f, $g \in \mathcal{A}(p)$ and $\alpha, \beta < 1$. If

$$\frac{f^{(p)}(\mathrm{z})}{p!} \in \mathcal{P}(\alpha), \quad \frac{g^{(p)}(\mathrm{z})}{p!} \in \mathcal{P}(\beta),$$

and

$$\psi^{(p-1)}(\mathrm{z}) = \int_0^{\mathrm{z}} \frac{\left(f^{(p-1)} * g^{(p-1)}\right)(t)}{t}dt,$$

then $\psi \in \mathcal{C}_p(p-1)$ provided that

$$(1-\alpha)(1-\beta) < \frac{3}{8(\ln 2-1)^2+4}.$$

The proof is simple by taking $z\psi^{(p)}(z) = \phi^{(p-1)}(z)$.

Theorem 7.2.11. Let $f,\ g,\ k \in \mathcal{A}(p)$, $\alpha,\ \beta,\ \gamma < 1$. If

$$\frac{f^{(p)}(z)}{p!} \in \mathcal{P}(\alpha),\ \frac{g^{(p)}(z)}{p!} \in \mathcal{P}(\beta),\ \frac{k^{(p)}(z)}{p!} \in \mathcal{P}(\gamma),$$

and

$$\Phi^{(p-1)}(z) = (f^{(p-1)} * g^{(p-1)} * k^{(p-1)})(z),$$

then $\Phi \in \mathcal{S}_p^*(p-1)$ provided

$$(1-\alpha)(1-\beta)(1-\gamma) < \frac{3}{\{8(\ln 2-1)^2+4\}\{(-4)(\ln 2-1)\}}.$$

Proof. By hypotheses on f, g and k and Lemma 2.10.15 we obtain

$$\begin{aligned}\Re\mathfrak{e}\left(\frac{F^{(p)}(z)}{p!} * \frac{h^{(p)}(z)}{p!}\right) &= \Re\mathfrak{e}\left(\frac{\Phi^{(p)}(z)+z\Phi^{(p+1)}(z)}{p!}\right)\\ &> 1-2(1-\alpha_1)(1-\beta),\end{aligned} \tag{7.2.30}$$

where $F^{(p-1)}(z) = \left(f^{(p-1)} * g^{(p-1)}\right)(z)$ and $\Re\mathfrak{e}\frac{F^{(p)}(z)}{p!} > \alpha_1$, $\alpha_1 = 1+4(1-\alpha)(1-\beta)(\ln 2-1)$. From (7.2.30) together with Lemma 2.10.14, we have

$$\Re\mathfrak{e}\left(\frac{\Phi^{(p)}(z)}{p!}\right) > 1-16(1-\alpha)(1-\beta)(1-\gamma)(\ln 2-1)^2, \qquad (z \in \mathbb{E}),$$

Using the same technique similar to that of Theorem 7.2.10, we get the required result. $\square$

Corollary 7.2.7. Let $f,\ g,\ k \in \mathcal{A}(p)$, $\alpha,\ \beta,\ \gamma < 1$. If

$$\frac{f^{(p)}(z)}{p!} \in \mathcal{P}(\alpha),\ \frac{g^{(p)}(z)}{p!} \in \mathcal{P}(\beta),\ \frac{k^{(p)}(z)}{p!} \in \mathcal{P}(\gamma),$$

and

$$\varphi^{(p-1)}(z) = (f^{(p-1)} * g^{(p-1)} * k^{(p-1)})(z),$$

then $\varphi \in \mathcal{C}_p(p-1)$ provided

$$(1-\alpha)(1-\beta)(1-\gamma) < \frac{3}{16(\ln 2-1)^2+8}.$$

For proving $\varphi(\in \mathcal{C}_p(p-1)$ it is sufficient to show that $\zeta^{(p-1)}(\mathrm{z}) = \mathrm{z}\varphi^{(p)}(\mathrm{z}) \in \mathcal{S}_p^*(p-1)$. By hypothesis on f, g, k and Lemma 2.10.15, we obtain

$$\begin{aligned}\mathfrak{Re}\left(\frac{\left(f^{(p)} * g^{(p)} * k^{(p)}\right)(\mathrm{z})}{p!}\right) &= \mathfrak{Re}\left(\frac{\zeta^{(p)}(\mathrm{z})+\mathrm{z}\zeta^{(p+1)}(\mathrm{z})}{p!}\right) \\ &> 1-4(1-\alpha)(1-\beta)(1-\gamma),\end{aligned}$$

and the proof is completed similarly to that of Theorem 7.2.10.

If $p=1$, then Theorem 7.2.10 and Corollary 7.2.11 were given in [43].

Theorem 7.2.12. Let $f_1, f_2, ..., f_\ell \in \mathcal{A}(p)$, $\alpha_1, \alpha_2, ..., \alpha_\ell < 1$. If

$$\frac{f_1^{(p)}(\mathrm{z})}{p!} \in \mathcal{P}(\alpha_1), \frac{f_2^{(p)}(\mathrm{z})}{p!} \in \mathcal{P}(\alpha_2), ..., \frac{f_\ell^{(p)}(\mathrm{z})}{p!} \in \mathcal{P}(\alpha_\ell),$$

and

$$\tau^{(p-1)}(\mathrm{z}) = \left(f_1^{(p-1)}(\mathrm{z}) * f_2^{(p-1)}(\mathrm{z}) * ... * f_\ell^{(p-1)}(\mathrm{z})\right), \text{ then } \tau \in \mathcal{S}^*(p-1), \tag{7.2.31}$$

provided

$$(1-\alpha_1)(1-\alpha_2)...(1-\alpha_\ell) < \frac{3}{\{8(\ln 2-1)^2+4\}\{(-4)(\ln 2-1)\}^{\ell-2}}, \quad \ell \geq 2. \tag{7.2.32}$$

Proof. For proving the above Theorem we use the principle of mathematical induction.

For $\ell = 2$ we have proved Theorem 7.2.10, thus (7.2.31) holds for $\ell = 2$.

Suppose that (7.2.31) hold true for $\ell = i$, that is

$$\tau^{(p-1)}(\mathrm{z}) = \left(f_1^{(p-1)}(\mathrm{z}) * f_2^{(p-1)}(\mathrm{z}) * ... * f_i^{(p-1)}(\mathrm{z})\right), \text{ then } \tau(\mathrm{z}) \in \mathcal{S}^*(p-1),$$

provided inequality (7.2.32) satisfied.

We have to prove that (7.2.31) holds true for $\ell = i+1$. For this consider

$$\tau^{(p-1)}(\mathrm{z}) = \left(f_1^{(p-1)}(\mathrm{z}) * f_2^{(p-1)}(\mathrm{z}) * \ldots * f_{i+1}^{(p-1)}(\mathrm{z})\right).$$

Now using given hypothesis on $f_j(\mathrm{z})$, $j = 1, 2, \ldots, i$ and Lemma 2.10.15, we have

$$\begin{aligned} \mathfrak{Re}\left(\frac{\varrho^{(p)}(\mathrm{z})}{p!} * \frac{f_{i+1}^{(p)}(\mathrm{z})}{p!}\right) &= \mathfrak{Re}\left(\frac{\tau^{(p)}(\mathrm{z}) + \mathrm{z}\tau^{(p+1)}(\mathrm{z})}{p!}\right) \\ &> 1-2(1-\alpha^*)(1-\alpha_{i+1}), \end{aligned} \tag{7.2.33}$$

where $\varrho^{(p-1)}(\mathrm{z}) = \left(f_1^{(p-1)}(\mathrm{z}) * f_2^{(p-1)}(\mathrm{z}) * \ldots * f_i^{(p-1)}(\mathrm{z})\right)$, and

$$\mathfrak{Re}\frac{\varrho^{(p)}(\mathrm{z})}{p!} > \alpha^*, \;\; \alpha^* = 1+4(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_i)(\ln 2-1)^{i-1}(-4)^{i-2}.$$

By using Lemma 2.10.14, from (7.2.33), we have

$$\mathfrak{Re}\frac{\tau^{(p-1)}(\mathrm{z})}{\mathrm{z}p!} > 1-8(1-\alpha^*)(1-\alpha_{i+1}).$$

Now with the same procedure as we have used in Theorem 7.2.10, we have $\tau(\mathrm{z}) \in \mathcal{S}_p^*(p-1)$ provided that

$$(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_i)(1-\alpha_{i+1}) < \frac{3}{\{8(\ln 2-1)^2+4\}\left\{(-4)(\ln 2-1)\right\}^{i-1}}.$$

Therefore the result is true for $\ell = i+1$ and hence, by using mathematical induction, (7.2.31) holds true for all $\ell \geq 2$. This completes the proof. □

7.3 Conclusion

In this chapter we have introduced a new subclass $\mathcal{R}_p$ of multivalent functions, and have studied some interesting properties by using convolution technique for this class $\mathcal{R}_p$. Also we point out some known consequences of our main results. Assigning particular values to different parameters in our main results several known results are deduced.

Chapter 8

Conclusion

The main objective of this work is to study certain aspects of Geometric Function Theory defined in an open unit disk $\mathbb{E} = \{\mathrm{z} : \mathrm{z} \in \mathbb{C}, |\mathrm{z}| < 1\}$. We have investigated and analyzed new classes of analytic spirallike, univalent and multivalent functions related with conic type regions. These classes are basically related with the convex, starlike uniformly convex and uniformly starlike functions, functions of bounded radius rotations and of bounded boundary rotations. Geometrical and analytical properties of these classes of functions have been studied.

Main techniques used to derive our results are convolution and differential subordination while classical approach is also exploited at some places to compact with the related problems. These techniques are used here and we study various properties including inclusion relations, radius problems, coefficients results and convolution properties. Some of these results are best possible and by giving different values to the parameters involved, we obtained some previously well known results.

This thesis consists of nine chapters, first seven chapters are summarized as follows; where as eighth and ninth chapters are on conclusion remarkes and references of the thesis respectively.

In first chapter, a brief introduction of Geometric Functions Theory, historical background and some recent advances are discussed.

In second chapter, basic results from Geometric Function Theory of a complex variable relevant to our work are presented. The classes $\mathcal{S}^*$ and $\mathcal{C}$ of starlike and convex functions are discussed respectively. These classes are defined as follows.

Let $f \in \mathcal{S}$. Then f maps $\mathbb{E}$ onto a starlike domain, iff

$$\mathfrak{Re}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} > 0, \quad \mathrm{z} \in \mathbb{E}.$$

Using the famous Alexander relation we have the class $\mathcal{C}$ of convex functions, that is

$$f \in C \quad \text{iff} \quad \mathrm{z}f' \in \mathcal{S}^*.$$

A generalization of starlike functions is the class of spirallike functions which was first introduced by Specek [118]. He defined these functions as follows.

A function f analytic in an open unit disk $\mathbb{E}$ is said to be in the class $\mathcal{S}_\beta$ of $\beta-$spirallike functions, iff

$$\mathfrak{Re}\left(e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right) > 0, \quad |\beta| < \frac{\pi}{2}, \quad \mathrm{z} \in \mathbb{E}. \tag{1}$$

If we take $\beta = 0$ in (1) then we have the class $\mathcal{S}^*$. The class of functions defined in (1) are the main motivation of our thesis. We generalized these functions in different directions and investigated some interesting properties. Specek showed that the functions defined in (1) are univalent in $\mathbb{E}$.

As generalization of univalent functions multivalent analytic functions have been discussed. The class $\mathcal{P}$ and its generalization with regard to the order λ $(0 \leq \lambda < 1)$ has been argued. The classes $\mathcal{V}_m$ and $\mathcal{R}_m$, $m \geq 2$, are reviewed. The concept of subordination followed by a short survey of conic domains and circular domains are included. Kanas and Wisniowska established and studied the conic domain Ω_k, $k \geq 0$. They defined the conic domains as follows. Let $k \in [0, \infty)$. For arbitrary chosen k, let Ω_k denote the following domain

$$\Omega_k = \left\{\mathfrak{u}+iv : \mathfrak{u} > k\sqrt{(\mathfrak{u}+1)^2 + v^2}\right\}.$$

The main tools of our work convolution and subordination are discussed. At the end of this chapter we gave some important known results which are used in the subsequent chapters. The results which are most important and throughout used in our work are given named as: Lemma 2.10.3, Lemma 2.10.5, Lemma 2.10.6, Lemma 2.10.10 and Lemma 2.10.13. These results provided necessary background for the successive chapters.

In third chapter, using the idea of spirallike functions and conic domains we have defined certain subclasses of generalized spirallike analytic functions as follows.

Definition 3.1. Let $f \in \mathcal{A}(p)$. Then $f \in k\text{–}\mathcal{UC}(p,\beta)$, for β real and $|\beta| < \frac{\pi}{2}$, iff

$$\left\{ \frac{e^{i\beta}}{p} \left(1 + \frac{\mathrm{z} f^{''}(\mathrm{z})}{f^{'}(\mathrm{z})} \right) \right\} \prec q_{k,\lambda}(\mathrm{z}) \quad (0 \leq \lambda < 1, \ \mathrm{z} \in \mathbb{E}) \,.$$

The corresponding class $k\text{–}\mathcal{UC}(p,\beta)$ is defined by Alexander type relation that is

$$f \in k\text{–}\mathcal{UC}(p,\beta) \qquad \text{iff} \qquad \frac{\mathrm{z} f^{'}}{p} \in k\text{–}\mathcal{UR}^{*}(p,\beta).$$

where $q_{k,\lambda}$ is given by (3.1.1).

Definition 3.2. For α, β real , $|\beta| < \frac{\pi}{2}$ and $-\frac{1}{2} \leq \gamma < 1$. Let $f \in \mathcal{A}(p)$ with $\frac{f^{'}(\mathrm{z}) f(\mathrm{z})}{p\mathrm{z}} \neq 0$ in $\mathbb{E}$ and

$$\pounds\left(\alpha,\beta,\gamma,f(\mathrm{z})\right) = \left(e^{i\beta} - \alpha\cos\beta\right) \frac{\mathrm{z} f^{'}(\mathrm{z})}{p f(\mathrm{z})} + \frac{\alpha\cos\beta}{p(1-\gamma)} \left(1 - \gamma + \frac{\mathrm{z} f^{''}(\mathrm{z})}{f^{'}(\mathrm{z})}\right).$$

Then

$$f \in k\text{–}\mathcal{UM}^{*}(p,\alpha,\beta,\gamma) \qquad \text{iff} \qquad \pounds\left(\alpha,\beta,\gamma,f(\mathrm{z})\right) \prec q_{k,\lambda} \text{ for } \mathrm{z} \in \mathbb{E}.$$

where $q_{k,\lambda}$ is given by (3.1.1).

These classes contain many well known ones for example, $\mathcal{S}^{*}$, $\mathcal{C}$, $\mathcal{M}(\alpha)$, $\mathcal{S}_{\beta}$, $\mathcal{C}_{\beta}$, $\mathcal{ST}$ and $\mathcal{UCV}$ the class of starlike, convex, $\alpha-$convex, spirallike, Robertson, uniformly starlike and uniformly convex functions respectively by varying parameters k, p, α, β and γ. Various interesting properties of newly defined classes such as necessary and sufficient condition, inclusion results, integral representation are discussed. Some important main results are stated as:

Result 3.1. Let $\alpha > 0$, $|\beta| < \frac{\pi}{2}$. Then

$$k\text{–}\mathcal{UM}^{*}(p,\alpha,\beta,0) \subset k\text{–}\mathcal{UR}^{*}(p,\beta).$$

Further
$$e^{i\beta}\frac{\mathrm{z}f'(\mathrm{z})}{pf(\mathrm{z})} \prec \tilde{q}_{k,\lambda}(\mathrm{z}) \prec q_{k,\lambda}(\mathrm{z}), \quad \mathrm{z}\in\mathbb{E},$$
where $\tilde{q}_{k,\lambda}(\mathrm{z})$ is the best dominant given by
$$\tilde{q}_{k,\lambda}(\mathrm{z}) = \left[\int_0^1 \left(\exp\int_t^{t\mathrm{z}} \frac{p_{k,\lambda}(u)-1}{u}du\right)dt\right]^{-1}.$$
In the above result we have proved inclusion relation between k–$\mathcal{UM}^*(p,\alpha,\beta,0)$ and k–$\mathcal{UR}^*(p,\beta)$. For particular values of parameters this result holds for the classes $\mathcal{M}(\alpha)$ and $\mathcal{S}^*$, that is, the class of Mocanu variation is contained in the class of starlike functions.

Result 3.2. Let $f\in\mathcal{A}(p)$. Then $f\in k$–$\mathcal{UM}^*(p,\alpha,\beta,\gamma)$, iff there exists a function $g\in k-UR^*(p,\beta)$ such that
$$g(\mathrm{z}) = f(\mathrm{z})\left(\frac{\mathrm{z}}{f(\mathrm{z})}\right)^{\frac{\alpha\cos\beta}{e^{i\beta}}}\left(f'(\mathrm{z})\right)^{\frac{\alpha\cos\beta}{e^{i\beta}(1-\gamma)}}.$$
This result is about the necessary and sufficiency criteria for the functions belonging to the class k–$\mathcal{UM}^*(p,\alpha,\beta,\gamma)$. Also we have established the relationship between the functions of the classes k–$\mathcal{UM}^*(p,\alpha,\beta,\gamma)$ and $k-UR^*(p,\beta)$.

Result 3.3. The class k–$\mathcal{UM}^*(p,0,\beta,\gamma)$ is preserved under the integral operator given in (3.2.14). Further
$$e^{i\beta}\frac{\mathrm{z}\mathfrak{I}'(\mathrm{z})}{p\mathfrak{I}(\mathrm{z})} \prec \widetilde{q}_{k,\lambda}(\mathrm{z}) \prec q_{k,\lambda}(\mathrm{z}), \quad \mathrm{z}\in\mathbb{E},$$
where $\widetilde{q}_{k,\lambda}(\mathrm{z})$ is the best dominant and is given by
$$\widetilde{q}_{k,\lambda}(\mathrm{z}) = \left[\int_0^1 \left(\exp\int_t^{t\mathrm{z}} \frac{q_{k,\lambda}(u)-1}{u}du\right)dt\right]^{-1} - \frac{\eta}{\frac{(\cos\beta+i\sin\beta)}{p}}.$$
In this result we have proved that the functions belonging to the class k–$\mathcal{UM}^*(p,0,\beta,\gamma)$ are closed under the Bernardi integral operator, defined in (3.2.14), and also find a best dominant function. For a special case, it means that the class of starlike and convex functions is preserved under the integral operator given by (3.2.14).

In fourth chapter, we introduced and examined some classes of spirallike functions using the generalized conic domain $\Omega_k[\mathbb{A},\mathbb{B}]$. These functions were first introduced by Sakaguchi [113]. We have extended this concept and introduced certain new classes as follows.

Definition 4.1. A function f $\in$S is said to be in the class $k-\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$, β is real, $|\beta| < \frac{\pi}{2}$, $k \geq 0$, $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$, iff

$$e^{i\beta}\frac{2\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})-f(-\mathrm{z})} \prec \cos\beta\frac{(\mathbb{A}+1)\,q_k(\mathrm{z})-(\mathbb{A}-1)}{(\mathbb{B}+1)\,q_k(\mathrm{z})-(\mathbb{B}-1)}+i\sin\beta, \quad \mathrm{z}\in\mathbb{E}.$$

Definition 4.2. Let $f \in \mathcal{S}$. Then $f \in k-\mathcal{UC}_s^{\beta}[\mathbb{A},\mathbb{B}]$, β is real, $|\beta| < \frac{\pi}{2}$, $k \geq 0, -1 \leq \mathbb{B} < \mathbb{A} \leq 1$, iff

$$e^{i\beta}\frac{2\left(\mathrm{z}f'(\mathrm{z})\right)'}{\left(f(\mathrm{z})-f(-\mathrm{z})\right)'} \prec \cos\beta\frac{(\mathbb{A}+1)\,q_k(\mathrm{z})-(\mathbb{A}-1)}{(\mathbb{B}+1)\,q_k(\mathrm{z})-(\mathbb{B}-1)}+i\sin\beta, \quad \mathrm{z}\in\mathbb{E}.$$

These newly defined classes map $\mathbb{E}$ onto generalized conic regions $\Omega_k[\mathbb{A},\mathbb{B}]$. The above classes contain many well-known classes for example $\mathcal{S}_s^*, \mathcal{C}_s, \mathcal{S}_s^*[\mathbb{A},\mathbb{B}], k-\mathcal{ST}_s, 0-\mathcal{UC}_s^0[1,-1]$, $\mathcal{C}_s^*[\mathbb{A},\mathbb{B}]$ and $k-\mathrm{UCV}_s$, by giving the specific values of parameters.

These classes are defined here and some of its properties such as inclusion result, Fekete-Szego inequality, necessary and sufficient condition are discussed. Some of these properties are stated as:

Result 4.1. Let $f \in k-\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$, $|\beta| < \frac{\pi}{2}$, $k \geq 0$ and $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$. Then the function

$$\varphi(\mathrm{z}) = \frac{1}{2}\left(f(\mathrm{z})-f(-\mathrm{z})\right),$$

belongs to $k-\mathcal{US}[\mathbb{A},\mathbb{B}]$ in E, where $k-\mathcal{US}[\mathbb{A},\mathbb{B}]$ is the class of Janowski starlike functions related with conic type region $k-\mathcal{P}[\mathbb{A},\mathbb{B}]$.

The above result indicates that the class $k-\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$ is a subclass of the class $\mathcal{K}$.

Result 4.2. Let $f \in k\text{--}\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$, $0 \leq k < \infty$, $|\beta| < \frac{\pi}{2}$, $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$. Then

$$\left|ua_2^2 - a_3\right| \leq \frac{1}{2}\begin{cases} \frac{\mathbb{A}-\mathbb{B}}{2}P_1(k)\left\{\frac{\mu\cos\beta e^{-i\beta}(\mathbb{A}-\mathbb{B})}{4}P_1(k) - \frac{(2D(k)-(\mathbb{B}+1)P_1(k))}{2}\right\}, & u > \delta_1, \\ \frac{\mathbb{A}-\mathbb{B}}{2}P_1(k), & \delta_1 \leq u \leq \delta_2, \\ \frac{\mathbb{A}-\mathbb{B}}{2}P_1(k)\left\{\frac{(2D(k)-(\mathbb{B}+1)P_1(k))}{2} - \frac{\mu\cos\beta e^{-i\beta}(\mathbb{A}-\mathbb{B})}{4}P_1(k)\right\}, & u < \delta_2, \end{cases}$$

where

$$\begin{aligned} \delta_1 &= \frac{2\left(2+2D(k) - (\mathbb{B}+1)P_1(k)\right)}{(\mathbb{A}-\mathbb{B})P_1(k)\cos\beta}e^{i\beta}, \\ \delta_1 &= \frac{2\left(2D(k) - (\mathbb{B}+1)P_1(k) - 2\right)}{(\mathbb{A}-\mathbb{B})P_1(k)\cos\beta}e^{i\beta}. \end{aligned}$$

and P_1, $D(k)$ are defined in (2.10.2) and (2.10.4).

In this result we are proved the Fekete-Szego inequality for the functions belonging to the class $k\text{--}\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$. As a special case, this result reduces to a result for $\mathcal{S}_s^*$.

Result 4.3. For $k \geq 0$, $|\beta| < \frac{\pi}{2}$, $-1 \leq \mathbb{B} < \mathbb{A} \leq 1$. A function $f \in k\text{--}\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$, iff

$$\frac{1}{\mathrm{z}}\left\{f(\mathrm{z}) * \left[\frac{\mathrm{z} - M_\beta \mathrm{z}^2}{(1-\mathrm{z})^2(1+\mathrm{z})}\right]\right\} \neq 0, \quad \mathrm{z} \in \mathbb{E}, \;\; 0 \leq \theta < 2\pi,$$

for all values of M_β where

$$M_\beta = \frac{\left(1+e^{-i\beta} i\sin\beta\right)\left((\mathbb{B}+1)q_k\left(e^{i\theta}\right) - (\mathbb{B}-1)\right) + e^{-i\beta}\cos\beta\left\{(\mathbb{A}+1)q_k\left(e^{i\theta}\right) - (\mathbb{A}-1)\right\}}{e^{-i\beta}\cos\beta\left\{(\mathbb{A}+1)q_k\left(e^{i\theta}\right) + (\mathbb{A}-1)\right\} - \left(1-e^{-i\beta} i\sin\beta\right)\left((\mathbb{B}+1)q_k\left(e^{i\theta}\right) + (\mathbb{B}-1)\right)},$$

The above result gives a necessary and sufficiency criteria for the functions belonging to the class $k\text{--}\mathcal{US}_s^{\beta}[\mathbb{A},\mathbb{B}]$. This result also holds for the class $\mathcal{S}_s^*$.

For suitable values of parameters the above results and newly defined classes are connected with the functions given in chapter 3. For example, in this chapter we have a result that the class $\mathcal{S}_s^*$ includes the class $\mathcal{C}$ and we are mentioned in chapter 3 that the class $k\text{--}\mathcal{UC}(p,\beta)$ contains the class $\mathcal{C}$ for $k = \beta = 0$ and $p = 1$.

In fifth chapter, some new subclasses of spirallike functions are discussed. These generalized spirallike analytic functions are defined as:

Definition 5.1. Let $f \in \mathcal{A}$ and be given by (2.1.1). Then, for β real and $|\beta| < \frac{\pi}{2}$, $f \in \mathcal{R}_m^* (\beta)$, iff

$$\left\{ e^{i\beta} \frac{\mathrm{z} f(\mathrm{z})'}{f(\mathrm{z})} \right\} \in \mathcal{P}_m \quad \mathrm{z} \in \mathbb{E},\ m \geq 2.$$

Definition 5.2. Let $f \in \mathcal{A}$ and be given by (2.1.1) and let, for $\frac{f(\mathrm{z})f'(\mathrm{z})}{\mathrm{z}} \neq 0$ in $\mathbb{E}$.

$$\mathcal{J}_m (\alpha, \beta, f(\mathrm{z})) = \left(e^{i\beta} - \alpha \cos\beta \right) \frac{\mathrm{z} f'(\mathrm{z})}{f(\mathrm{z})} + \alpha \cos\beta \left(1 + \frac{\mathrm{z} f''(\mathrm{z})}{f'(\mathrm{z})} \right).$$

Then

$$f \in \mathcal{M}_m^* (\alpha, \beta) \ \text{ iff } \mathcal{J}_m (\alpha, \beta, f(\mathrm{z})) \in \mathcal{P}_m, \quad \mathrm{z} \in \mathbb{E},\ \alpha, \beta \text{ real and } \ |\beta| < \frac{\pi}{2}.$$

Definition 5.3. Let $f \in \mathcal{A}$ and be given by (2.1.1) with $\frac{f(\mathrm{z})f'(\mathrm{z})}{\mathrm{z}} \neq 0$ in $\mathbb{E}$, and

$$\tilde{\mathcal{J}} (\alpha, \beta, \gamma, f(\mathrm{z})) = \left(e^{i\beta} - \alpha \cos\beta \right) \frac{\mathrm{z} f'(\mathrm{z})}{f(\mathrm{z})} + \frac{\alpha \cos\beta}{(1-\gamma)} \left(1 - \gamma + \frac{\mathrm{z} f''(\mathrm{z})}{f'(\mathrm{z})} \right),$$

for real $\alpha, \beta, |\beta| < \frac{\pi}{2}$ and $-\frac{1}{2} \leq \gamma < 1$. Then

$$f \in \mathcal{B}_m(\alpha, \beta, \gamma) \ \text{ iff } \ \tilde{\mathcal{J}} (\alpha, \beta, \gamma, f) \in \mathcal{P}_m \text{ for } \mathrm{z} \in \mathbb{E}, \quad m \geq 2.$$

Assigning suitable values to parameters the above classes generalized various known classes. Some of these known classes are $\mathcal{R}_m, \mathcal{S}^*, \mathcal{M}_\alpha$ and $\mathcal{S}_\beta(\lambda)$, the class of bounded radius rotation, the class of starlike functions, the class of functions with bounded Mocanu variation and the class of $\beta-$spirallike functions of order λ respectively.

Many interesting properties of these classes such as inclusion results, radius problem, necessary and sufficient condition, arc length problem, integral representation and integral preserving property are investigated and many known results are deduced from our main results as special cases. Some of main results are stated as follows.

Result 5.1. Let $\alpha > 0, |\beta| < \frac{\pi}{2}$, $m \geq 2$. Then $\mathcal{M}_m^*(\alpha,\beta) \subset \mathcal{R}_m^*(\beta)$.

The above result shows the relationship between $\mathcal{M}_m^*(\alpha,\beta)$ and $\mathcal{R}_m^*(\beta)$. In particular for $m = 2, \alpha = 1$ and $\beta = 0$, we have $\mathcal{C} \subset \mathcal{S}^*$.

Result 5.2. Let $f \in \mathcal{A}$. Then $f \in \mathrm{B}_m(\alpha,\beta,\gamma)$, $\alpha \neq 0$, iff, there exists a function $g \in \mathcal{B}_m(0,\beta,\gamma) = \mathcal{R}_m^*(\beta)$ such that

$$f(\mathrm{z}) = \left[\varsigma \int_0^{\mathrm{z}} t^{\varsigma-1} \left(\frac{g(t)}{t}\right)^{\frac{(1-\gamma)e^{i\beta}}{\alpha\cos\beta}} dt\right]^{\frac{1}{\varsigma}},$$

where

$$\varsigma = 1+\frac{(1-\gamma)(e^{i\beta}-\alpha\cos\beta)}{\alpha\cos\beta}.$$

In this result we have given the integral representation of the functions belonging to the class $\mathrm{B}_m(\alpha,\beta,\gamma)$ and also showed the relation between the functions of the class $\mathrm{B}_m(\alpha,\beta,\gamma)$ and $\mathcal{R}_m^*(\beta)$. As a special case the above result reduces to the Alexander relation between $\mathcal{C}$ and $\mathcal{S}^*$.

Result 5.3. Let $f \in \mathcal{B}_m(\alpha,\beta,\gamma)$, $\alpha > 0$ and let $L_{\mathrm{r}}(f)$ denote the length of the curve C,

$$C = f\left(\mathrm{r}e^{i\theta}\right), \quad 0 \leq \theta \leq 2\pi, \quad \text{and} \quad M(\mathrm{r}) = \max_{0\leq\theta\leq 2\pi}\left|f\left(\mathrm{r}e^{i\theta}\right)\right|.$$

Then, for $0 < \mathrm{r} < 1$,

$$L_{\mathrm{r}}(f) \leq \frac{\pi M(\mathrm{r})}{\alpha\cos\beta}\begin{cases} \left[m\left(1+\left|\alpha\cos\beta-e^{i\beta}\right|\right)+\frac{2\alpha\gamma\cos\beta}{1-\gamma}\right], & 0 < \alpha < 2, \\ m\left(1+\sqrt{\alpha(\alpha-2)\cos^2\beta+1}\right)+\frac{2\alpha\gamma\cos\beta}{1-\gamma}, & \alpha \geq 2. \end{cases}$$

In this result we have find the length of the curve for the functions belonging to the class $\mathcal{B}_m(\alpha,\beta,\gamma)$.

For particular values of parameters our newly defined classes were coincide with k–$\mathcal{U}\mathrm{R}^*(p,\beta)$, k–$\mathcal{UC}(p,\beta)$ and k–$\mathcal{UM}^*(p,\alpha,\beta,\gamma)$ defined in chapter 3. For example the class $\mathcal{M}_2^*(\alpha,0)$ defined here coincides with the class 0–$\mathcal{UM}^*(1,\alpha,0,0)$, of Mocanu variation defined in chapter 3.

In sixth chapter, spirallike functions are studied by using the operator $\mathbb{k}_p$ with the concept of reciprocal order. Using the idea of reciprocal order we defined the class $\mathrm{S}\mathcal{C}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$ as follows.

Definition 6.1. An analytic multivalent function $f(\mathrm{z}) \in \mathrm{S}\mathcal{C}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$, iff

$$\mathfrak{Re}\left\{\frac{2e^{i\beta}}{\mathfrak{b}}\left(\frac{\mathbb{k}_p(\mathfrak{a}+1,\mathfrak{c})f(\mathrm{z})}{\mathbb{k}_p(\mathfrak{a},\mathfrak{c})f(\mathrm{z})}-1\right)\right\} < (\lambda-1)\cos\beta,$$

where $\mathfrak{b} \in \mathbb{C}\diagdown\{0\}$, β is real with $|\beta| < \frac{\pi}{2}$, $\lambda > 1$.

This class gave several known classes as special cases, for example $\mathrm{S}\mathcal{C}_1^{0}(1,2,1,\lambda) = N(\lambda)$ and $\mathrm{S}\mathcal{C}_1^{0}(2,1,1,\lambda) = M(\lambda)$ are the basic subclasses of reciprocal order. The coefficient bound, sufficiency criteria, Fekete-Szego inequality and inclusion result are investigated here. Some of our main results are stated as:

Result 6.1. If $f \in \mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$, then

$$|a_2| \leq \frac{|\mathfrak{b}|\,|\eta|\,\mathfrak{c}}{p},$$

and

$$|a_{\ell+p-1}| \leq \frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(\ell+p-2)\phi_{\ell-1}(\mathfrak{a})}\prod_{j=1}^{\ell-2}\left(1+\frac{\mathfrak{a}\,|\mathfrak{b}|\,|\eta|}{(p+j)}\right), \quad \ell \geq 3,$$

where $\phi_{\ell-1}(\delta)$ is given by (2.9.2) and

$$\eta = (1-\lambda)\cos\beta + i\sin\beta.$$

In this result we find the bound of nth coefficient for the functions belonging to the class $\mathcal{SC}_p^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. The above result generalized many known results, see [89].

Result 6.2. Let $f\in$A(p) and satisfies

$$\sum_{\ell=1}^{\infty}\left|\left(1+\frac{2\ell}{\mathfrak{a}\mathfrak{b}}\right)e^{i\beta}-\lambda\cos\beta\right|\phi_{\ell}(\mathfrak{a})\left|a_{\ell+p}\right|<\left|\lambda\cos\beta-e^{i\beta}\right|. \tag{6.3.8}$$

Then $f\in\mathcal{SC}_{p}^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$.

The above result shows the sufficiency criteria for the functions belonging to the class $\mathcal{SC}_{p}^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. For suitable values assigning to parameters this result has many special cases, see [89, 16].

Result 6.3. Let $f\in\mathcal{SC}_{p}^{0}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$. Then

$$\left|a_{p+1}a_{p+3}-a_{p+2}^{2}\right|\leq\left[\frac{4\left|\mathfrak{b}\right|\mathfrak{c}(\mathfrak{c}+1)(1-\lambda)}{(p+1)(\mathfrak{a}+1)}\right]^{2}.$$

In this result we have found the upper bound of second Hankel determinant for the functions belonging to the class $\mathcal{SC}_{p}^{0}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$.

If we replace λ with $\frac{1}{\lambda}$ in the class $\mathrm{S}\mathcal{C}_{p}^{\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\lambda)$ then this class coincides with various classes defined in chapter 3, chapter 4 and chapter 5. For example $\mathrm{S}\mathcal{C}_{1}^{0}(1,2,1,\frac{1}{\lambda})=\mathcal{R}_{2}^{*}(0)=0\text{–}\mathcal{U}\mathrm{R}^{*}(1,0)=\mathcal{S}^{*}$ discussed in chapter 3 and chapter 5. All the work done here is new and connected with the previously known literature of the subject for specific ranges of parameters.

In seventh chapter, using pth derivative, we introduced and studied a subclass of multivalent analytic functions $\mathcal{R}_{p}$ as follows.

Definition 7.1. A function $f\in\mathcal{A}(p)$ given in (2.4.1) is said to belong to the class $\mathcal{R}_{p}$, if

$$\mathfrak{Re}\left(\frac{f^{(p)}(\mathrm{z})+\mathrm{z}f^{(p+1)}(\mathrm{z})}{p!}\right)>0,\qquad(\mathrm{z}\in E,\ p\in\mathbb{N}=\{1,2,...\}), \tag{2}$$

For $p=1,$ (2) can be written as $\mathfrak{Re}\left(\mathrm{z}f'(\mathrm{z})\right)'>0$ and this implies that $\mathfrak{Re}f'(\mathrm{z})>0$. This shows that f is univalent. We have studied and investigated some interesting properties of multivalent functions by means of convolution. Some of these properties are stated as follows.

Result 7.1. If $f \in \mathcal{A}(p)$ and

$$\mathfrak{Re}\left(\frac{f^{(p)}(\mathrm{z})+\mathrm{z}f^{(p+1)}(\mathrm{z})}{p!}\right) > -\frac{1}{4}, \qquad (\mathrm{z} \in \mathbb{E}), \tag{3}$$

then $f \in \mathcal{S}_p^*(p{-}1)$.

In this result we have proved that under the condition (3) (weak hypothesis), f is starlike of order $(p{-}1)$. For $p = 1$ the above result is proved in [117].

Result 7.2. If f and g belong to $\mathcal{R}_p$, and

$$h^{(p-1)}(\mathrm{z}) = f^{(p-1)}(\mathrm{z}) * g^{(p-1)}(\mathrm{z}),$$

then $h \in \mathcal{C}_p(p{-}1)$.

Ruscheweyh and Sheil [110] have proved that the class $\mathcal{C}$ is closed with respect to Hadamard product. In the above result we have extended the idea of [110] and proved that the Hadamard product of functions of the class $\mathcal{R}_p$ belong to the class $\mathcal{C}_p(p{-}1)$.

Result 7.3. Let $f_1, f_2, ..., f_\ell \in \mathcal{A}(p)$, $\alpha_1, \alpha_2, ..., \alpha_\ell < 1$. If

$$\frac{f_1^{(p)}(\mathrm{z})}{p!} \in \mathcal{P}(\alpha_1), \frac{f_2^{(p)}(\mathrm{z})}{p!} \in \mathcal{P}(\alpha_2), ..., \frac{f_\ell^{(p)}(\mathrm{z})}{p!} \in \mathcal{P}(\alpha_\ell),$$

and

$$\tau^{(p-1)}(\mathrm{z}) = \left(f_1^{(p-1)}(\mathrm{z}) * f_2^{(p-1)}(\mathrm{z}) * ... * f_\ell^{(p-1)}(\mathrm{z})\right),$$

then $\tau \in \mathcal{S}^*(p{-}1)$, provided

$$(1{-}\alpha_1)(1{-}\alpha_2)...(1{-}\alpha_\ell) < \frac{3}{\{8(\ln 2{-}1)^2{+}4\}\,\{(-4)(\ln 2{-}1)\}^{\ell-2}}, \quad \ell \geq 2.$$

In this result we have proved the general formula on the property of convolution of any finitely many functions $f_1^{(p-1)}(\mathrm{z}) * f_2^{(p-1)}(\mathrm{z}) * ... * f_\ell^{(p-1)}(\mathrm{z})$.

For $p = 1$, Lashin [43] have proved the above property for two and three functions, but here we generalized this result for nth functions. We have also proved that if $\mathfrak{Re} f^{(p)}(\mathrm{z}) > 0$, then f is multivalent. Multivalent starlike and multivalent convex functions are studied in chapter 3 and chapter 6, so this work is connected with previously mentioned chapters. All the results obtained in this chapter are new.

Chapter 9

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